# SOLVED PAPER

## M.Sc. MATHEMATICS (Part II)

## MATHEMATICAL STATISTICS

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# PAPER A/2016

# MATHEMATICAL STAT

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A/2016

## Q1(a)

If A and B are any two events and $P(B) \neq 1$, then prove that

$$P(A/\bar{B}) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

**Solved by Labeeb Ahmad**

**Solution:-** Since $P(A/B) = \frac{P(A \cap B)}{P(B)}$

Replace $B$ by $\bar{B}$

$$P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} \quad \text{--- (1)}$$

Also $A \cup \bar{A} = S$

$$P(A \cup \bar{A}) = P(S)$$

$$P(A) + P(\bar{A}) = 1$$

$$P(\bar{A}) = 1 - P(A) \quad \text{—— (2)}$$

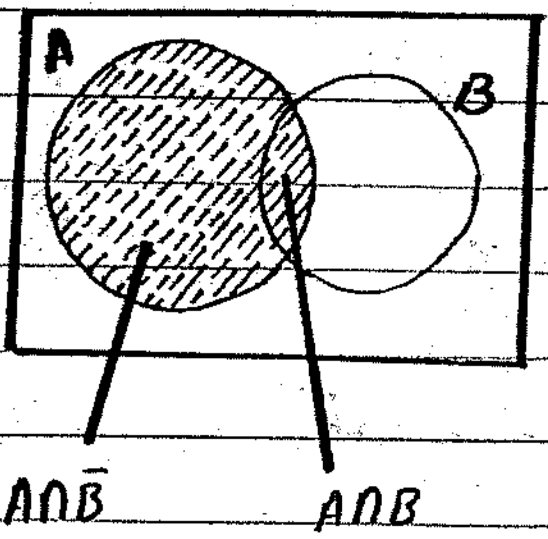
And

$$A = A \cap [B \cup \bar{B}]$$

$$A = (A \cap B) \cup (A \cap \bar{B})$$

$$P(A) = P[(A \cap B) \cup (A \cap \bar{B})]$$

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$




$$P(A) - P(A \cap B) = P(A \cap \bar{B})$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B) \quad \text{—— (3)}$$

Putting the value of (2) and (3) in (1)

we get

$$P(A/\bar{B}) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

Hence prove.

Solved by
Labeeb Ahmad

# A/2016

## Q1(b)

The probability that a doctor correctly diagnoses a particular illness is 0.7. Given that the doctor makes an incorrect diagosis, the probability that the patient files a lawsuit is 0.9. What is the probability that the doctor makes an incorrect diagnosis and patient sues?

## Solution:-

Let A = Correct diagnoses

$P(A) = 0.7$ (given in question)

Also let B = Doctor makes an incorrect diagonosis

$P(B) = 1 - P(A) = 1 - 0.7$

$= 0.3$

$P(A|B) = 0.9$ (given in question)

**Solved by Labeeb Ahmad**

As we know that

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Or $P(A/B) \cdot P(B) = P(A \cap B)$

$P(A \cap B) = (0.9) \times (0.3)$

$= 0.27$

**Solved by**
**Labeeb Ahmad**

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## Q2(a) For the Poisson Distribution with parameter m, show that

$$\mu_{r+1} = m\left[r\mu_{r-1} + \frac{d\mu_r}{dm}\right]$$

## Solution :-

$\mu_r = E(x-m)^r$ 'rth' moment about mean

By Definition

$$\mu_r = E(x-m)^r = \sum_{x=0}^{\infty} (x-m)^r \frac{e^{-m} m^x}{x!}$$

$$\mu_r = \sum_{x=0}^{\infty} \frac{1}{x!}\left[\{(x-m)^r\}\{e^{-m} m^x\}\right]$$

**Solved by Labeeb Ahmad**

Diff w. r. t "m"

$$\frac{d\mu_r}{dm} = \sum_{x=0}^{\infty} \frac{1}{x!}\Big[ (x-m)^r \{e^{-m} \cdot x m^{x-1} + m^x e^{-m}(-1)\}$$

$(x-m)^r$ ↓ Ist as it; $\{\ldots\}$ ↓ Derivation of 2nd

$$+ (e^{-m} m^x)\{r(x-m)^{r-1}(-1)\}\Big]$$

$(e^{-m} m^x)$ ↓ 2nd as it; $\{\ldots\}$ ↓ Derivation of Ist

$$\frac{d\mu_r}{dm} = \sum_{x=0}^{\infty} \frac{1}{x!}(x-m)^r e^{-m} m^x \left[\frac{x}{m} - 1\right]$$

$$- r \sum_{x=0}^{\infty} \frac{1}{x!}(x-m)^{r-1} e^{-m} . m^x$$

$$\frac{d\mu_r}{dm} = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} (x-m)^r . \left[\frac{(x-m)}{m}\right]$$

$$- r\mu_{r-1}$$

$$\frac{d\mu_r}{dm} = \frac{1}{m} \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} (x-m)^{r+1} - r\mu_{r-1}$$

$$\frac{d\mu_r}{dm} = \frac{1}{m}\mu_{r+1} - r\mu_{r-1}$$

**Solved by Labeeb Ahmad**

$$\frac{1}{m}\mu_{r+1} = r\mu_{r-1} + \frac{d\mu_r}{dm}$$

$$\mu_{r+1} = m\left[r\mu_{r-1} + \frac{d\mu_r}{dm}\right]$$

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**Q2(b)** Prove that if X and Y are independent Gamma variates, with parameters $l$ and $m$ respectivly, then $\frac{X}{Y}$ is a $\beta_2(l, m)$ variates.

**Solution** :- $X \sim \gamma_l$

$$f(x) = \frac{1}{\Gamma(l)} e^{-x} x^{l-1}$$

and $Y \sim \gamma_m$

$$f(y) = \frac{1}{\Gamma(m)} \cdot e^{-y} \cdot y^{m-1}$$

$\therefore$ The joint Probability of $x$ and $y$ is given by

$$f(x,y) = \left[\frac{e^{-x} x^{l-1}}{\Gamma(l)}\right]\left[\frac{1}{\Gamma(m)} \cdot e^{-y} \cdot y^{m-1}\right]$$

$$= \frac{1}{\Gamma(l)\Gamma(m)} e^{-(x+y)} x^{l-1} y^{m-1}$$

Solved by
Labeeb Ahmad

$U = \frac{X}{Y}$ and also suppose $V = Y$

$U = \frac{X}{V} \Rightarrow X = UV$ and $Y = V$

Now we transform $X, Y$ into two new variable $U$ and $V$ there for we use Jacobian of transformation.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

So $F(u,v) = f(x,y) \cdot |J|$

$$F(u,v) = \left[ \frac{1}{\Gamma(l)\,\Gamma(m)} e^{-(uv+v)} (uv)^{l-1} \cdot v^{m-1} \right] \cdot v$$

$$= \left[ \frac{1}{\Gamma(l)\,\Gamma(m)} e^{-v(u+1)} u^{l-1} \cdot v^{l-1} \cdot v^{m-1} \cdot v \right]$$

$$\left[ \frac{1}{\Gamma(l)\,\Gamma(m)} \cdot e^{-v(1+u)} u^{l-1} v^{l+m-1} \right]$$

Now the probability of "u" is

$$F(u) = \int_0^\infty F(u,v) \cdot dv$$

Solved by
Labeeb Ahmad

$$F(u) = \int_0^\infty \frac{e^{-v(1+u)} u^{l-1} v^{l+m-1}}{\Gamma(l)\,\Gamma(m)} \cdot dv$$

$$F(u) = \frac{u^{l-1}}{\Gamma(l)\Gamma(m)} \int_0^\infty e^{-v(1+u)} v^{l+m-1} \, dv \quad \text{---} (1)$$

Let $v(1+u) = \theta$

Solved by
Labeeb Ahmad

$$v = \frac{\theta}{1+u}$$

$$dv = \frac{d\theta}{1+u}$$

$$\begin{cases} \theta \to 0 & \text{when} \quad v \to 0 \\ \theta \to \infty & \text{when} \quad v \to \infty \end{cases}$$

substituting these in (1)

$$F(u) = \frac{u^{l-1}}{\Gamma(l)\Gamma(m)} \int_0^\infty e^{-\theta} \left[\frac{\theta}{1+u}\right]^{l+m-1} \cdot \left[\frac{d\theta}{1+u}\right]$$

$$= \frac{u^{l-1}}{\Gamma(l)\Gamma(m)} \int_0^\infty e^{-\theta} \theta^{l+m-1} \frac{d\theta}{(1+u)^{l+m-1+1}}$$

$$= \frac{u^{l-1}}{\Gamma(l)\Gamma(m)} \cdot \frac{1}{(1+u)^{l+m}} \int_0^\infty e^{-\theta} \theta^{(l+m)-1} \, d\theta$$

$$= \frac{u^{l-1}}{\Gamma(l)\Gamma(m)} \frac{1}{(1+u)^{l+m}} \Gamma(l+m)$$

$$F(u) = \frac{\Gamma(l+m)}{\Gamma(l)\,\Gamma(m)} \; \frac{u^{l-1}}{(1+u)^{l+m}}$$

$$F(u) = \frac{1}{\beta(l,m)} \; \frac{u^{l-1}}{(1+u)^{l+m}}$$

It is a density function of $\beta_2$ - distribution.

Hence it shows that $v = x/y$ is $\beta_2(l,m)$ variate.

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## Q3(a)

**Solved by Labeeb Ahmad**

A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 4 pins will be defective, what is the approximate probability that a box will fail to meet the guaranted qulity?

**Solution :-** let $p$ be the probability of the product being defective.

Then $p = \frac{5}{100} = 0.05$ and $n = 100$

The mean of the possion distribution

$\mu = np = 0.05 \times 100 = 5$

and the Possion distribution then becomes

$$P(x; 5) = \frac{e^{-5} \cdot (5)^x}{x!}, \quad x = 0, 1, 2 \ldots$$

and $e^{-5} = 0.0067$

We seek the probability that a box will fail to meet the guaranteed quality i-e the probability of there being more than 4 defective pins in a box of 100 pins. In other words we are to find $P(x > 4)$

Thus $P(x > 4) = 1 - P(x \le 4) = 1 - \sum_{x=0}^{4} \frac{e^{-5} \cdot 5^x}{x!}$

**Solved by**
**Labeeb Ahmad**

$$= 1 - e^{-5}\left[1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right]$$

$$= 1 - 0.0067[1 + 5 + 12.5 + 20.83 + 26.04]$$

$$= 1 - 0.0067[65.37]$$

$$= 1 - 0.4380$$

$$= 0.5620$$

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# Q3(b)

Prove that for Hyper-Geometric distribution, the following relation hold

$$P(X=x+1) = \frac{(k-x)(n-x)}{(x+1)(N-k-n+x+1)} P(X=x)$$

**Proof :-**

Take R.H.S

$$= \frac{(k-x)(n-x)}{(x+1)(N-k-n+x+1)} P(X=x)$$

Solved by Labeeb Ahmad

$$= \frac{(k-x)(n-x)}{(x+1)(N-k-n+x+1)} \left[\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}\right]$$

$$= \frac{(k-x)(n-x)}{(x+1)(N-k-n+x+1)} \left[\frac{\frac{k!}{x!(k-x)!} \times \frac{(N-k)!}{(n-x)!(N-k-n+x)!}}{{}^{N}C_{n}}\right]$$

$$= \frac{\cancel{(k-x)}\cancel{(n-x)}}{(x+1)(N-k-n+x+1)} \left[\frac{\frac{k!}{x!\cancel{(k-x)}(k-x-1)!} \times \frac{(N-k)!}{\cancel{(n-x)}(n-x-1)!(N-k-n+x)!}}{{}^{N}C_{n}}\right]$$

$$= \frac{\left[\dfrac{k!\,(N-k)!}{(x+1)(N-k-n+x+1)\cdot x!\,[k-(x+1)]!\,[n-(x+1)]!\,[N-k-n+x]!}\right]}{{}^{N}C_n}$$

$$= \frac{\left[\dfrac{k!}{(x+1)\cdot x!\,[k-(x+1)]!} \times \dfrac{(N-k)!}{[n-(x+1)]!\,(N-k-n+x+1)(N-k-n+x)!}\right]}{{}^{N}C_n}$$

$$= \frac{\dfrac{k!}{(x+1)!\,[k-(x+1)]!} \times \dfrac{(N-k)!}{[n-(x+1)]!\,[N-k-[n-(x+1)]]!}}{{}^{N}C_n}$$

$$= \frac{\binom{k}{x+1}\binom{N-k}{n-(x+1)}}{\binom{N}{n}}$$

$$= P(X = x+1)$$

$= L.H.S.$ Hence prove

Solved by
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## Q4(a) Prove that the mean deviation of the normal distribution is approximatly 4/5 of its standard deviation

## Solution :-

$$M.D = E\left[\,|x-\mu|\,\right]$$

$$M.D = \int_{-\infty}^{\infty} |x-\mu| \cdot \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \cdot dx$$

Put $\frac{x-\mu}{\sigma} = z$

$$dx = \sigma\, dz$$

$$\begin{cases} z \to \infty & \text{when } x \to \infty \\ z \to -\infty & \text{when } x \to -\infty \end{cases}$$

$$M.D = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z|\, e^{-z^2/2} (\sigma \cdot dz)$$

Solved by
Labeeb Ahmad

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z|\, e^{-z^2/2} \cdot dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{0} (-z)\, e^{-z^2/2} \cdot dz \int_{0}^{\infty} z\, e^{-z^2/2} \cdot dz$$

$$M\text{-}D = \frac{\sigma}{\sqrt{2\pi}} \int_0^\infty z e^{-z^2/2} dz + \int_0^\infty z e^{-z^2/2} dz$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty z e^{-z^2/2} . dz$$

$$= -\frac{\sqrt{2}}{\sqrt{\pi}} \sigma \int_0^\infty e^{-z^2/2} (-z) . dz$$

$$= -\sqrt{\frac{2}{\pi}} \sigma \left[ e^{-z^2/2} \right|_0^\infty$$

$$= -(0.7979)\sigma \left[ e^{-\infty} - e^{0} \right]$$

Since $(0.7979 \approx 4/5)$

$$= -\frac{4}{5} \sigma \left[ \frac{1}{e^{\infty}} - 1 \right]$$

$$= -\frac{4}{5} \sigma \left[ \frac{1}{\infty} - 1 \right]$$

$$= -\frac{4}{5} \sigma \left[ 0 - 1 \right]$$

$$= -\frac{4}{5} \sigma (-1)$$

$$= \frac{4}{5} \sigma \qquad \text{Hence prove}$$

Solved by
Labeeb Ahmad

**4(b)** Two caplets are selected at random from a bottle containing three aspirin two sedative and four laxative caplets. If x and Y are respectively, the number of aspirin and sedative caplets included among the two caplets drawn from the bottle. Find the probabilities associated with all possible pair of values of x and Y.

**(i)** Find the joint probability distribution of x and Y.

**(ii)** Find the marginal distribution of x and Y.

Solved by Labeeb Ahmad

## Solution :-

| X Aspirin | Y Sedative | Z Laxative | Total |
|---|---|---|---|
| 3 | 2 | 4 | 9 |

Number of items are selected = 2

Total sample space = $^9C_2 = 36$

And (0,0), (0,1), (0,2), (1,0), (1,1), (2,0) are possible pairs.

$$P(0,0) = \frac{\binom{3}{0}\binom{2}{0}\binom{4}{2}}{\binom{9}{2}} = \frac{1\cdot 1\cdot 6}{36} = \frac{6}{36}$$

$$P(0,1) = \frac{\binom{3}{0}\binom{2}{1}\binom{4}{1}}{\binom{9}{2}} = \frac{1\cdot 2\cdot 4}{36} = \frac{8}{36}$$

$$P(0,2) = \frac{\binom{3}{0}\binom{2}{2}\binom{4}{0}}{\binom{9}{2}} = \frac{1\cdot 1\cdot 1}{36} = \frac{1}{36}$$

$$P(1,0) = \frac{\binom{3}{1}\binom{2}{0}\binom{4}{1}}{\binom{9}{2}} = \frac{3\cdot 1\cdot 4}{36} = \frac{12}{36}$$

$$P(1,1) = \frac{\binom{3}{1}\binom{2}{1}\binom{4}{0}}{\binom{9}{2}} = \frac{3\cdot 2\cdot 1}{36} = \frac{6}{36}$$

$$P(2,0) = \frac{\binom{3}{2}\binom{2}{0}\binom{4}{0}}{\binom{9}{2}} = \frac{3\cdot 1\cdot 1}{36} = \frac{3}{36}$$

(i) The joint probability of x and Y

| X \ Y | 0 | 1 | 2 | g(x) |
|---|---|---|---|---|
| 0 | 6/36 | 8/36 | 1/36 | 15/36 |
| 1 | 12/36 | 6/36 | 0 | 18/36 |
| 2 | 3/36 | 0 | 0 | 3/36 |
| h(y) | 21/36 | 14/36 | 1/36 | 1 |

(ii) Marginal distribution of X

| x | 0 | 1 | 2 |
|---|---|---|---|
| g(x) | 15/36 | 18/36 | 3/36 |

**Solved by Labeeb Ahmad**

Marginal distribution of Y

| Y | 0 | 1 | 2 |
|---|---|---|---|
| h(y) | 21/36 | 14/36 | 1/36 |

# Section II

A/2016

## Q5 (a)

$$R_{2.31} = \sqrt{\frac{r_{23}^2 + r_{12}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2}}$$

**Proof:-**

$$R_{2.31} = \frac{Cov(x_2, \hat{x}_2)}{\sqrt{Var(x_2) \cdot Var(\hat{x}_2)}}$$

$$= \frac{\sum x_2 \cdot \hat{x}_2 / n}{\sqrt{\frac{\sum x_2^2}{n} \cdot \frac{\sum (\hat{x}_2)^2}{n}}}$$

$$= \frac{\sum x_2 \hat{x}_2 / n}{\frac{1}{n}\sqrt{\sum x_2^2 \cdot \sum (\hat{x}_2)^2}}$$

Solved by
Labeeb Ahmad

$$= \frac{\sum x_2 \cdot \hat{x}_2}{\sqrt{\sum x_2^2 \cdot \sum (\hat{x}_2)^2}}$$

Now $\sum x_2 \cdot \hat{x}_2 = \sum x_2 (x_2 - x_{2.31})$

since ($\because \hat{x}_2 = x_2 - x_{2.31}$)

$$= \sum x_2^2 - \sum x_2 \cdot x_{2.31}$$

$$= \sum x_2^2 - \sum x_{2.31} \cdot x_{2.31} \quad (\because \sum x_2 \cdot x_{2.31} = \sum x_{2.31} x_{2.31})$$

$$= n(S_2^2 - S_{2.31}^2)$$

where $S_{2.31}^2$ is the sample variance of residual

Also $\sum(\hat{x}_2)^2 = \sum(x_2 - x_{2.31})^2$

$$= \sum(x_2^2) + \sum x_{2.31}^2 - 2\sum x_2 \cdot x_{2.31}$$

$$= \sum x_2^2 + \sum x_{2.31}^2 - 2\sum x_{2.31}^2$$

$$= \sum x_2^2 - \sum x_{2.31}^2$$

$$= n\frac{\sum x_2^2}{n} - n\frac{\sum x_{2.31}^2}{n}$$

$$= n(S_2^2 - S_{2.31}^2)$$

And $\sum x_2^2 = n S_2^2$

Substituting these values in the formula we get

$$R_{2.31} = \frac{S_2^2 - S_{2.31}^2}{S_2\sqrt{S_2^2 - S_{2.31}^2}} = \left[1 - \frac{S_{2.31}^2}{S_2^2}\right]^{1/2}$$

Squaring both side

**Solved by**
**Labeeb Ahmad**

$$R_{2.31}^2 = 1 - \frac{S_{2.31}^2}{S_2^2}$$

The quantity $S_{2.31}^2$ can be expressed in term of the simple co-relation co-efficient b/w the pairs of the variables as below

A/2016, A/2012, A/2011, A/2003

## Q5(b) State and prove Central Limit theorem.

## Solution :-

**Statment :-** If $X_1, X_2, X_3 \dots X_n$ Constitute a random sample from an infinite Population with mean "$\mu$", variance "$\sigma^2$", and the moment generating function $M_x(t)$ Then the Limiting distribution of

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

as $n \to \infty$ is the standard normal Distribution.

## Proof

let $Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

$$M_z(t) = E(e^{tz})$$

$$= E\left[e^{t\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right]}\right]$$

Solved by
Labeeb Ahmad

$$= E\left[e^{\frac{\sqrt{n}\,t\,(\bar{x}-\mu)}{\sigma}}\right]$$

Now Taking ln both sides

$$\ln M_Z(t) = \ln\left[\left(e^{-\frac{\mu\sqrt{n}t}{\sigma}}\right)\left(M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n\right]$$

$$= \ln e^{-\frac{\mu\sqrt{n}t}{\sigma}} + \ln\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

$$= -\frac{\mu\sqrt{n}t}{\sigma}\ln e + n\ln\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]$$

$$= -\frac{\mu\sqrt{n}t}{\sigma} + n\ln\left[1 + \mu_1'\left(\frac{t}{\sigma\sqrt{n}}\right) + \frac{\mu_2'}{2!}\left(\frac{t}{\sigma\sqrt{n}}\right)^2 + \frac{\mu_3'}{3!}\left(\frac{t}{\sigma\sqrt{n}}\right)^3 + \cdots\cdots\right]$$

$$= \frac{-\mu\sqrt{n}t}{\sigma} + n\ln\left[1 + \left\{\mu_1'\left(\frac{t}{\sigma\sqrt{n}}\right) + \frac{\mu_2'}{2}\left(\frac{t^2}{\sigma^2 n}\right) + \frac{\mu_3'}{6}\left(\frac{t^3}{\sigma^3 n\sqrt{n}}\right) + \cdots\right\}\right]$$

**Solved by**
**Labeeb Ahmad**

As $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots$

$$= \frac{-\mu\sqrt{n}t}{\sigma} + n\left\{\left[\mu_1'\left(\frac{t}{\sigma\sqrt{n}}\right) + \frac{\mu_2'}{2}\left(\frac{t^2}{\sigma^2 n}\right) + \cdots\right]\right.$$

$$\left. - \frac{1}{2}\left[\mu_1'\left(\frac{t}{\sigma\sqrt{n}}\right) + \frac{\mu_2'}{2}\left(\frac{t^2}{\sigma^2 n}\right) + \cdots\right]^2 + \cdots\cdots\right\}$$

$$\ln M_Z(t) = \frac{-\mu\sqrt{n}\,t}{\sigma} + \frac{n\mu_1' t}{\sigma\sqrt{n}} + \frac{\not{n} t^2 \mu_2'}{2\sigma^2 \not{n}} + \cdots$$

$$-\frac{n}{2}\left[\frac{\mu_1'^2 t^2}{\sigma^2 n} + \frac{\mu_2'^2 t^4}{4\sigma^4 n^2} + \cdots\right] + \cdots$$

$\because \mu = \mu_1'$

$$= -\frac{\mu_1'\sqrt{n}t}{\sigma} + \frac{\mu_1'\sqrt{n}t}{\sigma} + \frac{\mu_2' t^2}{2\sigma^2} - \frac{\mu_1'^2 t^2}{2\sigma^2} - \frac{\mu_2'^2 t^4}{2\sigma^4 n} - \cdots$$

$$= (\mu_2' - \mu_1'^2)\frac{t^2}{2\sigma^2} - \frac{\mu_2'^2 t^4}{2\sigma^4 n} - \cdots$$

$$= \frac{\sigma^2 t^2}{2\sigma^2} - \frac{\mu_2'^2 t^4}{2\sigma^4 n} - \cdots$$

$$= \frac{t^2}{2} - \frac{\mu_2'^2 t^4}{2\sigma^4 n} - \cdots$$

**Solved by**
**Labeeb Ahmad**

Taking Lim $n \longrightarrow \infty$

$$\lim_{n\to\infty} \ln M_Z(t) = \lim_{n\to\infty}\left[\frac{t^2}{2} - \frac{\mu_2'^2 t^4}{2\sigma^4 n} - \cdots\right]$$

$$= t^2/2 - 0 - 0 - 0$$

$$= t^2/2$$

Taking Antilog$_e$ both side $M_Z(t) = e^{t^2/2}$ which is m.g.f of standard normal Distribution.

A/2016



# Q6(a) Given the joint density

$$f(x,y) = \begin{cases} 2 & \text{for } 0<y<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

Show that $\mu_{Y/X} = \frac{x}{2}$ and $\mu_{X/Y} = \frac{1+Y}{2}$

## Solution :-

$$g(x) = \int_{-\infty}^{\infty} f(x,y) \cdot dy$$

**Solved by Labeeb Ahmad**

$$= \int_{0}^{x} 2 \cdot dy$$

$$= 2 [y|_{0}^{x}$$

$$g(x) = 2x$$

Now $f(Y/x) = \frac{f(x,y)}{g(x)}$

$$= \frac{2}{2x} = \frac{1}{x}$$

$$\mu_{Y/x} = \int_{0}^{x} y \cdot f(Y/x) \cdot dy$$

$$= \int_{0}^{x} y \cdot \frac{1}{x} \cdot dy$$

$$= \frac{1}{x} \cdot \left[\frac{y^2}{2}\right]_{0}^{x} \Rightarrow \frac{1}{2x} \cdot x^2$$

$$\boxed{\mu_{Y/X} = x/2}$$

Now $h(y) = \int_{-\infty}^{\infty} f(x,y)\,.\,dx$

$$= \int_{y}^{1} f(x,y)\,.\,dx$$

$$= \int_{y}^{1} 2\,.\,dx$$

$$= 2\left[x\right]_{y}^{1}$$

$$= 2(1-y)$$

$$f(x/y) = \frac{f(x,y)}{h(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

$$\mu_{x/y} = \int_{y}^{1} x\,.\,f(x,y)\,.\,dx$$

$$= \int_{y}^{1} x\,.\,\frac{1}{(1-y)}\,.\,dx$$

**Solved by Labeeb Ahmad**

$$= \frac{1}{1-y}\left[\frac{x^2}{2}\right]_{y}^{1}$$

$$= \frac{1}{2(1-y)}\,.\,(1-y^2)$$

$$= \frac{(1-y)(1+y)}{2(1-y)}$$

$$\boxed{\mu_{x/y} = \frac{1+y}{2}}$$

A/2016



**Q6(b)** If X and Y are two uncorrelated variables if $U = X+Y$ and $V = X-Y$, find the co-rrelation between U and V in terms of $\sigma_x$ and $\sigma_y$ where $\sigma_x$ and $\sigma_y$ are standard deviation of X and Y.

## Solution :-

$$r_{UV} = \frac{Cov(U,V)}{\sqrt{Var(U) \cdot Var(V)}}$$

Solved by
Labeeb Ahmad

$$r_{UV} = \frac{E\left[(U-\bar{U})(V-\bar{V})\right]}{\sqrt{E(U-\bar{U})^2 \, E(V-\bar{V})^2}}$$

put $U = X+Y$ and $V = X-Y$

$$r_{UV} = \frac{E\left[\{(X+Y)-\overline{(X+Y)}\}\{(X-Y)-\overline{(X-Y)}\}\right]}{\sqrt{E\left[(X+Y)-\overline{(X+Y)}\right]^2 E\left[(X-Y)-\overline{(X-Y)}\right]^2}}$$

Solved by
Labeeb Ahmad

$$r_{UV} = \frac{E\left[\{X+Y-\bar{X}-\bar{Y}\}\{X-Y-\bar{X}+\bar{Y}\}\right]}{\sqrt{E\left[X+Y-\bar{X}-\bar{Y}\right]^2 E\left[X-Y-\bar{X}+\bar{Y}\right]^2}}$$

$$r_{UV} = \frac{E\left[\{(X-\bar{X})+(Y-\bar{Y})\}\{(X-\bar{X})-(Y-\bar{Y})\}\right]}{\sqrt{E\left[(X-\bar{X})+(Y-\bar{Y})\right]^2 E\left[(X-\bar{X})-(Y-\bar{Y})\right]^2}}$$

$$r_{UV} = \frac{E\left[(x-\bar{x})^2 - (x-\bar{x})(y-\bar{y}) + (y-\bar{y})(x-\bar{x}) - (y-\bar{y})^2\right]}{\sqrt{E\left((x-\bar{x})^2 + (y-\bar{y})^2 + 2\left[(x-\bar{x})(y-\bar{y})\right]\right) \cdot E\left((x-\bar{x})^2 + (y-\bar{y})^2 - 2\left[(x-\bar{x})(y-\bar{y})\right]\right)}}$$

$$= \frac{E\left[(x-\bar{x})^2 - (y-\bar{y})^2\right]}{\sqrt{\left[E(x-\bar{x})^2 + E(y-\bar{y})^2 + 2E(x-\bar{x})(y-\bar{y})\right]\left[E(x-\bar{x})^2 + E(y-\bar{y})^2 - 2E(x-\bar{x})(y-\bar{y})\right]}}$$

Since X and Y are uncorrelated

$$Cov(X,Y) = E(x-\bar{x})(y-\bar{y}) = 0$$

$$= \frac{E(x-\bar{x})^2 - E(y-\bar{y})^2}{\sqrt{\left(E(x-\bar{x})^2 + E(y-\bar{y})^2\right)^2}}$$

$$= \frac{\sigma_x^2 - \sigma_y^2}{E(x-\bar{x})^2 + E(y-\bar{y})^2}$$

**Solved by**
**Labeeb Ahmad**

$$r_{UV} = \frac{\sigma_x^2 - \sigma_y^2}{\sigma_x^2 + \sigma_y^2}$$

# Q7(a) If the joint probability density of $x_1$ and $x_2$ is given by

$$f(x_1,x_2) = \begin{cases} e^{-(x_1+x_2)} & \text{for } x_1 > 0 \text{ and } x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find (i) the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_1}{X_1+X_2}$

(ii) the marginal density of $Y_2$

## Solution

$$Y_2 = \frac{X_1}{X_1+X_2} \quad \text{and} \quad Y_1 = X_1 + X_2$$

$$Y_2 = \frac{X_1}{Y_1} \Rightarrow X_1 = Y_1 Y_2$$

Now
$$Y_1 = X_1 + X_2$$
$$Y_1 = Y_1 Y_2 + X_2$$
$$X_2 = Y_1 - Y_1 Y_2$$
$$X_2 = Y_1(1 - Y_2)$$

Solved by Labeeb Ahmad

$$J\frac{(X_1, X_2)}{(Y_1, Y_2)} = \begin{vmatrix} \dfrac{\partial X_1}{\partial Y_1} & \dfrac{\partial X_1}{\partial Y_2} \\ \dfrac{\partial X_2}{\partial Y_1} & \dfrac{\partial X_2}{\partial Y_2} \end{vmatrix}$$

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 1-Y_2 & -Y_1 \end{vmatrix}$$

$= -Y_1Y_2 - Y_1(1-Y_2)$

$= -Y_1Y_2 - Y_1 + Y_1Y_2$

$= -Y_1$

Solved by Labeeb Ahmad

$|J| = |-Y_1| = Y_1$

$f(Y_1, Y_2) = f(x_1, x_2) \cdot |J|$

$f(Y_1, Y_2) = e^{-Y_1} \cdot Y_1 . \quad Y_1 > 0$

This is a joint probability density function.

(ii)

The marginal density of $Y_2$

$f(Y_2) = \int_0^\infty f(Y_1, Y_2) \cdot dY_1$

$= \int_0^\infty e^{-Y_1} \cdot Y_1 \cdot dY_1$

$= \int_0^\infty e^{-Y_1} \cdot Y_1^{2-1} \cdot dY_1$

$= \Gamma(2) = 1! = 1$

$f(Y_2) = 1$

## Q7(b) If X has the standard normal Distribution, find the probability of $X^2$.

## Solution :-

$$X \sim N(0,1)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad -\infty < x < \infty$$

$$df = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

let $y = x^2$

$$x = \sqrt{y}$$

$$dx = \frac{1}{2\sqrt{y}} dy$$

Then the distribution of $y$ is

$$df = \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{dy}{2\sqrt{y}} \quad \text{——} (1)$$

Now put $\frac{y}{2} = \theta$

Solved by Labeeb Ahmad

$$y = 2\theta$$

$$dy = 2\,d\theta$$

So (1) becomes

$$df = \frac{1}{\sqrt{2\pi}} e^{-\theta} \cdot \frac{1}{x\sqrt{2\theta}} \cdot x \cdot d\theta$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{\pi}} e^{-\theta} \theta^{-1/2} \cdot d\theta\right]$$

$$\therefore = \frac{1}{2}\left[\frac{1}{\Gamma(1/2)} \cdot e^{-\theta} \theta^{\frac{1}{2}-1}\right]$$

$$0 \leq \theta < \infty$$

This is a Gamma variable with parameter 1/2.

* ———————————— *

**Solved by**
**Labeeb Ahmad**

Labeeb Ahmad
From Sharaqpur Sharif
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Solved by
Labeeb Ahmad

A/2016, A/2012, A/2008, A/2004

**Q8(a)** Derive chi-square distribution as a limiting form of F-distribution.

## Solution:-

Let F-distribution with n and m degree of freedom.

$$df(F) = \frac{(n/m)^{\frac{n}{2}}\, F^{\frac{n}{2}-1}}{\beta\left(\frac{n}{2}, \frac{m}{2}\right)\left(1+\frac{n}{m}F\right)^{\frac{n+m}{2}}}\, dF$$

$$= \frac{\Gamma\left(\frac{n}{2}+\frac{m}{2}\right)\,(n)^{n/2}\, F^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)(m)^{n/2}\left(1+\frac{n}{m}F\right)^{\frac{n+m}{2}}}\, dF$$

$$= \frac{n^{\frac{n}{2}}\, F^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{n+m}{2}\right)}{m^{n/2}\,\Gamma\left(\frac{m}{2}\right)\left(1+\frac{nF}{m}\right)^{\frac{n+m}{2}}}\, dF \quad (1)$$

Take $\dfrac{\Gamma\left(\frac{n+m}{2}\right)}{m^{n/2}\,\Gamma\left(\frac{m}{2}\right)\left(1+\frac{nF}{m}\right)^{\frac{n+m}{2}}}$

**Solved by Labeeb Ahmad**

As we know Stirling's approximation

$$\Gamma(n+1) = n! \approx e^{-n}\, n^{n+\frac{1}{2}}\sqrt{2\pi}$$

Using the approximation

$$= \frac{e^{-\left\{\frac{n+m}{2}-1\right\}} \left[\frac{n+m}{2}-1\right]^{\frac{n+m}{2}-1+\frac{1}{2}} \sqrt{2\pi}}{m^{n/2} \cdot e^{-\left\{\frac{m}{2}-1\right\}} \left[\frac{m}{2}-1\right]^{\frac{m}{2}-1+\frac{1}{2}} \sqrt{2\pi} \left(1+\frac{nF}{m}\right)^{\frac{n+m}{2}}}$$

$$= \frac{e^{-\frac{n}{2}}\, e^{-\frac{m}{2}+1} \left(\frac{n+m-2}{2}\right)^{\frac{n+m}{2}-\frac{1}{2}}}{m^{n/2}\, e^{-\frac{m}{2}+1} \left(\frac{m-2}{2}\right)^{\frac{m}{2}-\frac{1}{2}} \left(1+\frac{nF}{m}\right)^{\frac{n+m}{2}}}$$

$$= \frac{e^{-\frac{n}{2}} \dfrac{(n+m-2)^{\frac{n+m-1}{2}}}{2^{\left(\frac{n+m-1}{2}\right)}}}{m^{n/2} \dfrac{(m-2)^{\frac{m-1}{2}}}{2^{\frac{m-1}{2}}} \left(1+\frac{nF}{m}\right)^{\frac{n+m}{2}}}$$

$$= \frac{e^{-\frac{n}{2}} \dfrac{(n+m-2)^{\frac{n+m-1}{2}}}{2^{\frac{n}{2}} \cdot 2^{\frac{m-1}{2}}}}{m^{n/2} \dfrac{(m-2)^{\frac{m-1}{2}}}{2^{\frac{m-1}{2}}} \left(1+\frac{nF}{m}\right)^{\frac{n+m}{2}}}$$

**Solved by Labeeb Ahmad**

$$= \frac{e^{-n/2}\,(n+m-2)^{\frac{n+m-1}{2}}}{m^{n/2}\cdot 2^{n/2}\,(m-2)^{\frac{m-1}{2}}\left(1+\frac{nF}{m}\right)^{\frac{n+m}{2}}}$$

$$= \frac{e^{-n/2}\, m^{\frac{n+m-1}{2}}\left[1+\frac{n-2}{m}\right]^{\frac{n+m-1}{2}}}{m^{n/2}\cdot 2^{n/2}\, m^{\frac{m-1}{2}}\left[1-\frac{2}{m}\right]^{\frac{m-1}{2}}\left(1+\frac{nF}{m}\right)^{\frac{n+m}{2}}}$$

$$= \frac{e^{-n/2}\, m^{\frac{n+m-1}{2}}\left[1+\frac{n-2}{m}\right]^{\frac{m}{2}}\cdot\left[1+\frac{n-2}{m}\right]^{\frac{n-1}{2}}}{m^{\frac{n+m-1}{2}}\left[1-\frac{2}{m}\right]^{\frac{m}{2}}\cdot\left(1-\frac{2}{m}\right)^{-1}\left(1+\frac{nF}{m}\right)^{\frac{n}{2}+\frac{m}{2}}}$$

$$= \frac{e^{-n/2}\left[\left(1+\frac{n-2}{m}\right)^{\frac{m}{n-2}}\right]^{\frac{n-2}{2}}\left(1+\frac{n-2}{m}\right)^{\frac{n-1}{2}}}{2^{n/2}\left(\left[1-\frac{2}{m}\right]^{-\frac{m}{2}}\right)^{-1}\left(1-\frac{2}{m}\right)^{-1}\left[1+\frac{nF}{m}\right]^{\frac{n}{2}}\left[\left(1+\frac{n}{m}F\right)^{\frac{m}{nF}}\right]^{\frac{nF}{2}}}$$

Applying $\lim_{m\to\infty}$

Solved by Labeeb Ahmad

$$= \frac{e^{-n/2}\, e^{\frac{n-2}{2}}\cdot(1)}{2^{n/2}\, e^{-1}\cdot(1)\cdot(1)\cdot e^{\frac{nF}{2}}}$$

$$= \frac{e^{-\frac{n}{2}} \cdot e^{\frac{n}{2}} \cdot \cancel{e^{-1}}}{2^{n/2} \cancel{e^{-1}} \; e^{(nF)/2}}$$

$$= \frac{1}{2^{n/2} \cdot e^{\frac{nF}{2}}}$$

Putting this in (1)

$$df(F) = \frac{n^{n/2} \; F^{\frac{n}{2}-1}}{\Gamma(n/2)} \cdot \frac{1}{2^{n/2} \cdot e^{\frac{nF}{2}}} \cdot dF \quad (2)$$

put $\chi^2 = nF$

$\frac{\chi^2}{n} = F$

and $\frac{d\chi^2}{n} = dF$

Solved by
Labeeb Ahmad

Put in (2)

$$df(F) = \frac{n^{n/2} \left(\frac{\chi^2}{n}\right)^{\frac{n}{2}-1}}{2^{n/2} \Gamma(n/2) \cdot e^{\frac{n}{2} \cdot \frac{\chi^2}{n}}} \cdot \frac{d\chi^2}{n}$$

$$= \frac{\cancel{n^{n/2}} (\chi^2)^{\frac{n}{2}-1}}{2^{n/2} \Gamma(n/2) \cancel{(n)^{\frac{n}{2}-1}} \cdot e^{\chi^2/2}} \cdot \frac{d\chi^2}{\cancel{n}}$$

$$= \frac{(\chi^2)^{\frac{n}{2}-1} \cdot e^{-\chi^2/2}}{2^{n/2} \Gamma(n/2)} \; d(\chi^2)$$

Hence Prove

## Q8(b) Stat and Prove Partioning Property of $\chi^2$-distribution.

**Statment :-** A $\chi^2$ r.v can be partioned into two or more than two parts which are also $\chi^2$-r.v and sum of degree of freedom equal to the total degree of freedom.

**Proof :-** suppose that $x_1, x_2, x_3 --- x_n$ be a random sample from a normal Population $N(\mu, \sigma^2)$. Then we can define the Sum of squares from the mean as:

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu)^2$$

Solved by
Labeeb Ahmad

$$= \sum_{i=1}^{n}\left[(x_i-\bar{x})+(\bar{x}-\mu)\right]^2$$

$$= \sum_{i=1}^{n}\left[(x_i-\bar{x})^2+(\bar{x}-\mu)^2+ 2(x_i-\bar{x})(\bar{x}-\mu)\right]$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2+(\bar{x}-\mu)^2\sum_{i=1}^{n}1 + 2(\bar{x}-\mu)\left\{\sum_{i=1}^{n}x_i-\bar{x}\sum_{i=1}^{n}1\right\}$$

$$\sum_{i=1}^{n}(x_i-\mu)^2=\sum_{i=1}^{n}(x_i-\bar{x})^2+n(\bar{x}-\mu)^2+2(\bar{x}-\mu)\left(\sum_{i=1}^{n}x_i-n\bar{x}\right)$$

$$\because \bar{x}=\frac{\sum x_i}{n}$$

$$\text{or} \quad n\bar{x}=\sum x_i$$

$$=\sum_{i=1}^{n}(x_i-\bar{x})^2+n(\bar{x}-\mu)^2+2(\bar{x}-\mu)(n\bar{x}-n\bar{x})$$

$$\sum_{i=1}^{n}(x_i-\mu)^2=\sum_{i=1}^{n}(x_i-\bar{x})^2+n(\bar{x}-\mu)^2$$

Dividing both side by $\sigma^2$

$$\sum_{i=1}^{n}\left[\frac{x_i-\mu}{\sigma}\right]^2=\sum_{i=1}^{n}\left[\frac{x_i-\bar{x}}{\sigma}\right]^2+n\left[\frac{\bar{x}-\mu}{\sigma}\right]^2$$

$$=\sum_{i=1}^{n}\left[\frac{x_i-\bar{x}}{\sigma}\right]^2+\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right]^2 \quad \text{——} (1)$$

Since $\left(\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2$ is $\chi_1^2$ r.v as the square of standard normal random variable has $\chi_1^2$-dist.

$\sum_{i=1}^{n}\left[\dfrac{x_i-\bar{x}}{\sigma}\right]^2$ is $\chi^2$-distribution with $n-1$ degree of freedom

Solved by
Labeeb Ahmad

Thus equation (1) given

$$\chi_n^2=\chi_{n-1}^2+\chi_1^2$$

**Q9(a)** Prove that student t-dist approaches to the standard normal distribution as $n \to \infty$

**Solution :-** The p.d.f of "t"-dist is

$$f(t) = \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2}, \frac{1}{2}\right)} \cdot \left[1+t^2/n\right]^{-\frac{1}{2}(n+1)}$$

$$= \frac{1}{\sqrt{n}} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)} \cdot \left[1+\frac{t^2}{n}\right]^{-\frac{1}{2}(n+1)}$$

$$= \frac{1}{\sqrt{n}\sqrt{\pi}} \underbrace{\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}}_{A} \cdot \underbrace{\left[1+t^2/n\right]^{-\frac{1}{2}(n+1)}}_{B}$$

$$f(t) = \frac{1}{\sqrt{n\pi}} A \cdot B$$

Solved by
Labeeb Ahmad

where $A = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$ and $B = \left(1+\frac{t^2}{2}\right)^{-\frac{1}{2}(n+1)}$

Take $A = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma(n/2)}$

As we know that by stirling's approximation

$$\Gamma(n+1) = n! \approx e^{-n}\, n^{n+\frac{1}{2}} \sqrt{2\pi}$$

So A becomes

$$A = \frac{e^{-\left[\frac{n}{2}+\frac{1}{2}-1\right]} \left(\frac{n}{2}-\frac{1}{2}\right)^{\frac{n}{2}-\frac{1}{2}+\frac{1}{2}} \sqrt{2\pi}}{e^{-\left[\frac{n}{2}-1\right]} \left(\frac{n}{2}-1\right)^{\frac{n}{2}-1+\frac{1}{2}} \sqrt{2\pi}}$$

$$A = \frac{e^{-\frac{n}{2}+1} \cdot e^{-1/2} \left[\frac{n}{2}-\frac{1}{2}\right]^{n/2}}{e^{-\frac{n}{2}+1} \left(\frac{n}{2}-1\right)^{\frac{n}{2}-\frac{1}{2}}}$$

$$A = \frac{e^{-1/2} \left[\frac{n-1}{2}\right]^{n/2}}{\left(\frac{n-2}{2}\right)^{\frac{n}{2}-\frac{1}{2}}}$$

Solved by
Labeeb Ahmad

$$A = \frac{e^{-1/2}\, 2^{-\frac{n}{2}}\, (n-1)^{n/2}}{2^{-\frac{n}{2}+\frac{1}{2}}\, (n-2)^{\frac{n}{2}-1/2}}$$

$$A = \frac{e^{-1/2}\, n^{n/2} \left[1-\frac{1}{n}\right]^{n/2}}{2^{1/2}\, n^{\frac{n}{2}-\frac{1}{2}} \left[1-\frac{2}{n}\right]^{\frac{n}{2}-\frac{1}{2}}}$$

$$A = \frac{e^{-1/2}\, n^{n/2} \left[1-\frac{1}{n}\right]^{n/2}}{\sqrt{2}\, n^{n/2} \cdot n^{-1/2} \left[1-\frac{2}{n}\right]^{\frac{n}{2}} \left[1-\frac{2}{n}\right]^{-\frac{1}{2}}} \quad \text{———} \ (2)$$

Now Take B

$$B = \left[1+\frac{t^2}{n}\right]^{-\frac{n}{2}-\frac{1}{2}}$$

$$B = \left[1+\frac{t^2}{n}\right]^{-\frac{n}{2}} \cdot \left[1+\frac{t^2}{n}\right]^{-\frac{1}{2}}$$

$$\therefore B = \left(\left[1+\frac{t^2}{n}\right]^{\frac{n}{t^2}}\right)^{-\frac{t^2}{2}} \left[1+\frac{t^2}{n}\right]^{-1/2} \quad (3)$$

Putting the value of A and B in (1)

$$f(t) = \frac{1}{\sqrt{n}\sqrt{\pi}} \frac{e^{-1/2}\left[1-\frac{1}{n}\right]^{n/2}}{\sqrt{2}\, n^{-1/2}\left[1-\frac{2}{n}\right]^{\frac{n}{2}}\left[1-\frac{2}{n}\right]^{-1/2}} \left[\left\{1+\frac{t^2}{n}\right\}^{\frac{n}{t^2}}\right]^{-\frac{t^2}{2}}\left(1+\frac{t^2}{n}\right)^{-1/2}$$

Now Apply $\lim_{n\to\infty}$

$$\lim_{n\to\infty} f(t)$$

$$= \frac{e^{-1/2} \lim_{n\to\infty}\left(\left[1-\frac{1}{n}\right]^{-n}\right)^{\frac{1}{2}} \lim_{n\to\infty}\left[\left\{1+\frac{t^2}{n}\right\}^{\frac{n}{t^2}}\right]^{-\frac{t^2}{2}} \lim_{n\to\infty}\left[1+\frac{t^2}{n}\right]^{-1/2}}{\sqrt{2\pi}\lim_{n\to\infty}\left[\left\{1-\frac{2}{n}\right\}^{-\frac{n}{2}}\right]^{-1} \lim_{n\to\infty}\left[1-\frac{2}{n}\right]^{-1/2}}$$

$$= \frac{e^{-1/2} \cdot e^{-1/2} \cdot e^{-\frac{t^2}{2}} \cdot (1)}{\sqrt{2\pi}\; e^{-1} \;(1)}$$

Solved by Labeeb Ahmad

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}}$$ it is a p.d.f of standard normal Distribution

## Q9(b) If $T \sim t_\nu$, then show that $T^2 \sim F(1,\nu)$

## Solution:-

As $T \sim t_\nu$

$$df(t) = \frac{1}{\sqrt{\nu}\,\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\left[1+\frac{t^2}{\nu}\right]^{-\frac{1}{2}[1+\nu]} dt$$

Put $t^2 = F$

$$t = \sqrt{F}$$

$$dt = \frac{1}{2\sqrt{F}}\,dF$$

$$df(F) = \frac{1}{\sqrt{\nu}\,\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\left[1+\frac{F}{\nu}\right]^{-\frac{1}{2}[1+\nu]} \frac{dF}{2\sqrt{F}}$$

$$= \frac{1}{2}\left[\frac{[1/\nu]^{1/2}}{\beta\left(\frac{1}{2},\frac{\nu}{2}\right)} F^{\frac{1}{2}-1} \cdot \frac{1}{\left[1+\frac{F}{\nu}\right]^{\left[\frac{1+\nu}{2}\right]}}\right]$$

$$= \frac{1}{2}\left(F(1,\nu)\right)$$

**Solved by**
**Labeeb Ahmad**

Hence $T^2$ follows $F(1,\nu)$.

where $\frac{1}{2}$ constant.

# PAPER A/2015

# MATHEMATICAL STAT

**Solved by**
**Labeeb Ahmad**

A/2015

## Q1(a) Find the Mean deviation of the Normal Distribution.

**Solution:** See paper {A/2016, Q4(a)} at page "14"

A/2015

## Q1(b) If the moment generating function of x is $M(t) = e^{166t + 200t^2}$

Find (i) $P(170 < X < 200)$ (ii) $P(148 < X < 172)$

**Solution:-**

Comparing $M(t) = e^{166t + 200t^2}$ with the m.g.f of $N(\mu, \sigma^2)$ i.e $M(t) = e^{\mu t + \frac{\sigma^2}{2}t^2}$

we find $\mu = 166$ and $\sigma^2 = 400$

To find the desired probabilities, we transform x values to z values.

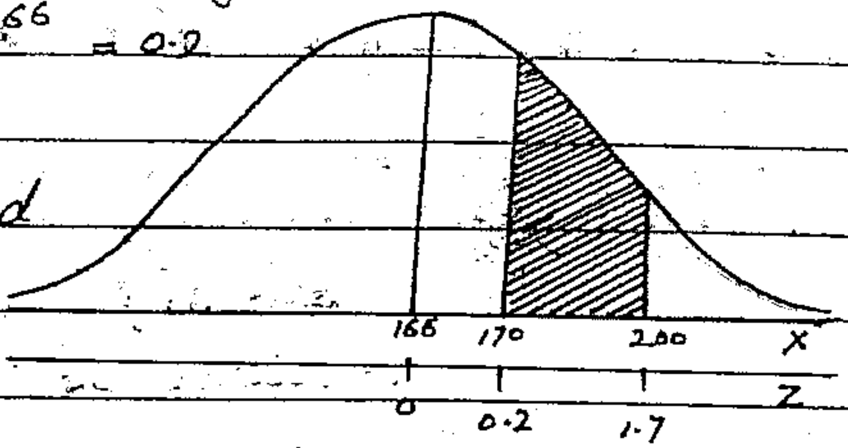
using $z = \frac{x - 166}{20}$, Therefor

**(i)** At $x = 170$, we get

$$Z = \frac{170-166}{20} = 0.2$$

At $x = 200$, we find

$$Z = \frac{200-166}{20} = 1.7$$




$P(170 < X < 200) = P(0.2 < Z < 1.7)$

$= P(0 < Z < 1.7) - P(0 < Z < 0.2)$

using table ⇒ $= 0.4554 - 0.0793$

$= 0.3761$

**Solved by**
**Labeeb Ahmad**

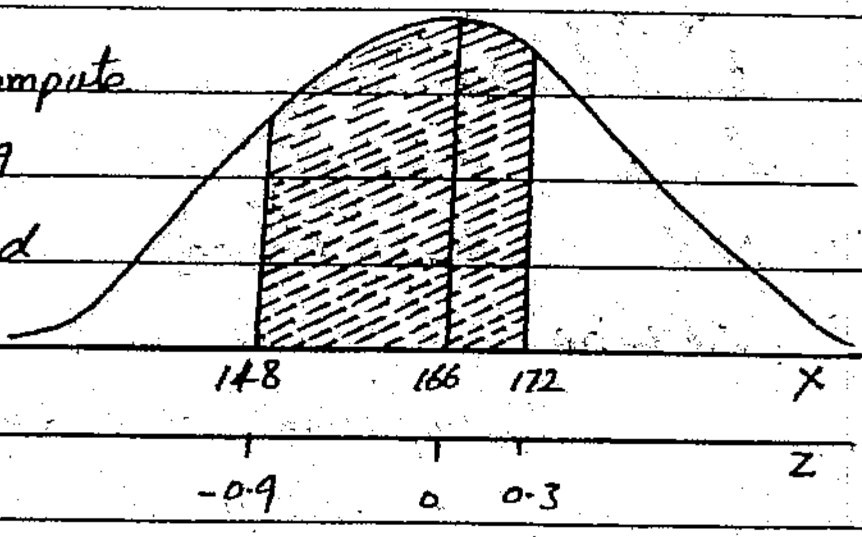
**(ii)**

At $X = 148$ we compute

$$Z = \frac{148-166}{20} = -0.9$$

For $X = 172$, we find

$$Z = \frac{172-166}{20} = 0.3$$




$P(148 \leq X \leq 170) = P[-0.9 < Z \leq 0.3]$

$= P[-0.9 < Z \leq 0] + P[0 \leq Z < 0.3]$

using Table

$= 0.3159 + 0.1179$

$= 0.4338$

## Q2(a)

Bag-I Contains 6 white and 5 black balls while Bag-II Contains 4 white and 3 black balls. One ball is drawn from Bag-I and replaced unseen in Bag-II. What is the probability that, a ball now drawn from the Bag II is (i) Black (ii) white

## Solution :-

**Solved by Labeeb Ahmad**

| | Bag-I | Bag-II |
|---|---|---|
| White | 6 | 4 |
| Black | 5 | 3 |
| Total | 11 | 7 |

If white ball is drawn from Bag-I and placed in Bag-II.
Then There will be 3 black and total 8 balls.

If black ball is drawn from Bag-I and placed in Bag-II
There will be 4 black and total 8 balls.

(i) P(Black ball is drawn from 2nd Bag)

$$= \frac{6}{11} \times \frac{3}{8} + \frac{5}{11} \times \frac{4}{8}$$

$$= \frac{18}{88} + \frac{20}{88} = \frac{38}{88}$$

$$= \frac{19}{44}$$

Now If black ball is drawn from Bag-I and placed in 2nd-Bag then there will be 4 white and total 8 balls.

If white ball is drawn from Bag-I and placed in 2nd-Bag then there will be 5 white and total 8 balls.

(ii) P(white ball is drawn from 2nd-Bag)

$$= \frac{6}{11} \times \frac{4}{8} + \frac{5}{11} \times \frac{5}{8}$$

$$= \frac{24}{88} + \frac{25}{88}$$

$$= \frac{49}{88}$$

Solved by
Labeeb Ahmad

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# Q2(b)

A continuous r.v has p.d.f

$$f(x) = \begin{cases} k(2-x)(x-5) & 2 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

Find Harmonic mean? Calculate distribution function and estimate the probabilities 1. $P(|x-\mu| \le 3\sigma)$ and $P(|x-\mu| \ge 3\sigma)$

**Solution** The function $f(x)$ will be p.d.f if $f(x) \ge 0$ for every $x$

and $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

**Solved by Labeeb Ahmad**

The first condition is satisfied when $K \ge 0$. The 2nd condition will be satisfied if

$$\int_{-\infty}^{\infty} f(x) \cdot dx = 1$$

i-e

$$\int_{-\infty}^{2} f(x) \cdot dx + \int_{2}^{5} f(x) \cdot dx + \int_{5}^{\infty} f(x) \cdot dx = 1$$

i-e

$$\int_{-\infty}^{2} 0 \cdot dx + \int_{2}^{5} k(2-x)(x-5) + \int_{5}^{\infty} 0 \cdot dx = 1$$

$$\int_{2}^{5} k(-x^2 + 7x - 10) = 1$$

$$-k\int_2^5 (x^2-7x+10)\,dx = 1$$

$$-k\left[\frac{x^3}{3}-\frac{7x^2}{2}+10x\Big|_2^5\right]=1$$

$$-k\left[\frac{5^3}{3}-\frac{7}{2}5^2+10(5)-\frac{2^3}{3}+\frac{7}{2}(2)^2-10(2)\right]=1$$

$$-k\left[\frac{125}{3}-\frac{175}{2}+50-\frac{8}{3}+14-20\right]=1$$

$$-k\left[-\frac{9}{2}\right]=1$$

**Solved by**
**Labeeb Ahmad**

$$\boxed{k=\frac{2}{9}}$$

Hence $f(x)=\begin{cases}\frac{2}{9}(2-x)(x-5) & 2\le x\le 5\\ 0 & ;\text{else where}\end{cases}$

Harmonic Mean

$H.M = 1/\frac{1}{H}$ where $\frac{1}{H}=E\left(\frac{1}{x}\right)$

$$\frac{1}{H}=E\left(\frac{1}{x}\right)=\int_{-\infty}^{\infty}\frac{1}{x}\,f(x)\,dx$$

$$=\int_2^5\left(\frac{1}{x}\right)\cdot\frac{2}{9}(2-x)(x-5)\cdot dx$$

$$=\frac{2}{9}\int_2^5\frac{1}{x}\left(-x^2+7x-10\right).dx$$

$$=\frac{2}{9}\int_2^5 -x+7-\frac{10}{x}\,.\,dx$$

$$1/H = \frac{2}{9}\left| -\frac{x^2}{2} + 7x - 10\ln x \right|_2^5$$

$$= \frac{2}{9}\left[ -\frac{25}{2} + 35 - 10\ln(5) + \frac{4}{2} - 14 + 10\ln 2 \right]$$

$$= \frac{2}{9}(1.3370)$$

$$\frac{1}{H} = 0.29713$$

$$H\text{-}M = \frac{1}{1/H} = \frac{1}{0.29713} = 3.3655$$

$$H.M = 3.365510$$

To compute the Distribution function we find $F(x) = P(X<x) = \int_{-\infty}^{x} f(x)\cdot dx$

For any $x$ such that $-\infty < x \leq 2$

$$F(x) = \int_{-\infty}^{2} 0 \cdot dx = 0$$

If $2 \leq x \leq 5$ we have

$$F(x) = \int_{-\infty}^{2} 0 \cdot dx + \int_{2}^{x} \frac{2}{9}(2-x)(x-5)\,dx$$

$$= 0 + \int_{2}^{x} \frac{2}{9}(-x^2 + 7x - 10)\,dx$$

**Solved by**
**Labeeb Ahmad**

$$= \frac{2}{9}\left[ -\frac{x^3}{3} + \frac{7x^2}{2} - 10x \right|_2^x ]$$

$$F(x) = \frac{2}{9}\left[-\frac{x^3}{3}+\frac{7}{2}x^2-10x+\frac{8}{3}-14+20\right]$$

$$F(x) = \frac{2}{9}\left[-\frac{x^3}{3}+\frac{7}{2}x^2-10x+\frac{26}{3}\right]$$

$$F(x) = \frac{-2x^3}{27}+\frac{7}{9}x^2-\frac{20}{9}x+\frac{52}{27}$$

Finally for $x > 5$ we have

$$F(x) = \int_{\infty}^{2} 0.dx + \int_2^5 \frac{2}{9}(2-x)(x-5)\,dx + \int_5^{\infty} 0.dx$$

$$= 1$$

Hence

$$F(x) = \begin{cases} 0 & \text{for } x < 2 \\ -\frac{2}{27}x^3+\frac{7}{9}x^2-\frac{20}{9}x+\frac{52}{27} & \text{for } 2 \le x \le 5 \\ 1 & \text{for } x > 5 \end{cases}$$

Now the mean of X is

$$\mu = E(X) = \frac{2}{9}\int_2^5 x(2-x)\cdot(x-5)\,dx$$

$$= \frac{2}{9}\int_2^5 (-x^3+7x^2-10x)\,dx$$

Solved by
Labeeb Ahmad

$$= \frac{2}{9}\left[-\frac{x^4}{4}+\frac{7x^3}{3}-\frac{10x^2}{2}\right]_2^5$$

$$= \frac{2}{9}\left[\left(-\frac{625}{4} + \frac{875}{3} - \frac{250}{2}\right) - \left(-4 + \frac{56}{3} - 20\right)\right]$$

$$= \frac{2}{9}\left[\frac{125}{12} + \frac{16}{3}\right] = \frac{2}{9}\left(\frac{189}{12}\right) = \frac{7}{2}$$

Also

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2, \text{ where}$$

$$\therefore E(X^2) = \int_2^5 x^2 f(x)\cdot dx = \frac{2}{9}\int_2^5 x^2(2-x)(x-5)\cdot dx$$

$$= \frac{2}{9}\int_2^5 (-x^4 + 7x^3 - 10x^2)\cdot dx$$

$$= \frac{2}{9}\left[-\frac{x^5}{5} + \frac{7x^4}{4} - \frac{10x^3}{3}\Big|_2^5\right]$$

$$= \frac{2}{9}\left[\left(-\frac{3125}{5} + \frac{4375}{4} - \frac{1250}{3}\right) - \left(-\frac{32}{5} + \frac{112}{4} - \frac{80}{3}\right)\right]$$

$$= \frac{2}{9}\left(\frac{1143}{20}\right) = \frac{127}{10}$$

$$\therefore Var(X) = \frac{127}{10} - \left(\frac{7}{2}\right)^2 = \frac{9}{20}$$

$$\sigma^2 = \frac{9}{20}$$

Solved by Labeeb Ahmad

$$\sigma = \sqrt{\frac{9}{20}} = 0.67082$$

Now $P(|X-\mu| < 3\sigma)$

$$= P(-3\sigma < X - \mu < 3\sigma)$$

$$= P(\mu - 3\sigma < X < \mu + 3\sigma)$$

$$= P\left(\frac{7}{2} - 3(0.67082) < x < \frac{7}{2} + 3(0.67082)\right)$$

$$= P(1.48754 < x < 5.51246)$$

$$P(1.48754 < x < 5.51246) = \int_{1.48754}^{2} 0 \cdot dx + \int_{2}^{5} \frac{2}{9}(2-x)(x-5) + \int_{5}^{5.51246} 0 \cdot dx$$

$$= 0 + \frac{2}{9}\int_{2}^{5}(-x^2 + 7x - 10)\,dx + 0$$

$$= \frac{2}{9}\left[\frac{-x^3}{3} + \frac{7x^2}{2} - 10x \Big|_2^5\right]$$

$$= \frac{2}{9}\left[\left(\frac{-125}{3} + \frac{175}{2} - 50\right) - \left(\frac{-8}{3} + 14 - 20\right)\right]$$

$$= \frac{2}{9}\left(\frac{9}{2}\right)$$

$$= 1$$

Also $P(|X-4| \geq 3\sigma) = 1 - P(|X-4| < 3\sigma)$

$$= 1 - 1$$

$$= 0$$

Solved by
Labeeb Ahmad

## Q3(a) There are three coins; identical in apperance. one of which is ideal and other two biased with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively for a head. One coin is taken at random and tossed twice. If a head appears both the times, what is the probablity that the ideal coin was chosen?

## Solution :-

Solved by
Labeeb Ahmad

Let

$A_1$ denote the event that the ideal coin is selected

$A_2$ denote the event that the 2nd coin is selected and

$A_3$ denote the event that the 3rd coin is selected

Then $P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$

Again ~~Let~~ $E$ denote the event that head appears both the times when the coin is tossed twice. Then

$P(E \backslash A_1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ (Probability of two heads given that ideal coin is tossed).

$P(E \backslash A_2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$P(E \backslash A_3) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$

Hence by Baye's Theorem, the desired probability is

$$P(A_1 \backslash E) = \frac{P(A_1)\, P(E \backslash A_1)}{\sum P(A_i)\, P(E \backslash A_i)} \qquad i = 1, 2, 3 \cdots$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{4}}{\frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{9} + \frac{1}{3} \cdot \frac{4}{9}} = \frac{1/12}{29/208}$$

$$= \frac{9}{29}$$ Answer

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Q3(b) Stat and Prove Bayes Theorm.

**Statment** :- If the events $A_1, A_2 \cdots A_k$ form a partion of sample space $S$ that is, the events $A_i$ are mutually exclusive and their union is $S$ and $B$ is any other event of $S$ such that

$$P(A_i|B) = \frac{P(A_i)\,P(B/A_i)}{\sum_{i=1}^{k} P(A_i)\,P(B/A_i)} \quad \text{for } i = 1,2,3 \cdots k$$

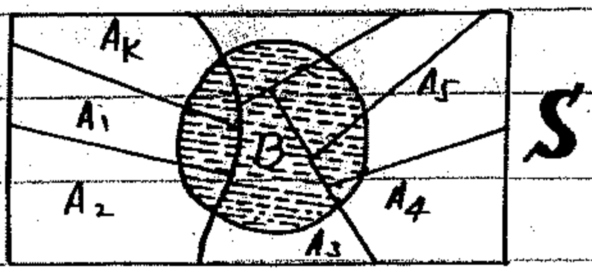
**Solved by Labeeb Ahmad**

**Proof** :- By the multiplicative law of probabilities, we have

$$P(B \cap A_i) = P(B)\,P(A_i/B)$$
$$= P(A_i)\,P(B/A_i)$$

Equating the equivalent relations of $P(B\cap A_i)$ and solving for $P(A_i/B)$, we get

$$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{P(B)}$$




B is shaded

We may write the events $B$ as

$B = S \cap B$ ( See the Venn Diagram)

$= (A_1 \cup A_2 \cup A_3 \text{ --- } A_k) \cap B$

$= (A_1 \cap B) \cup (A_2 \cap B) \cup \text{ - - - - } (A_k \cap B)$

where the $A_i \cap B$ are also mutually exclusive.

Solved by
Labeeb Ahmad

There for

$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \text{ - - - } P(A_k \cap B)$

using the multiplication law of probabilities, we may express each term $P(A_i \cap B)$ as $P(A_i) P(B|A_i)$ Then

$$P(B) = P(A_1) P(B|A_1) + P(A_2) P(B|A_2) + \text{ --- --- } P(A_k) P(B|A_k)$$

$$= \sum_{i=1}^{k} P(A_i) P(B|A_i)$$

This result is generally known as the theorem on total probability. Replacing $P(B)$ by the total probability formula for the event $B$, we obtain Bay's theorem.

$$P(A_i/B) = \frac{P(A_i) P(B/A_i)}{\sum_{i=1}^{k} P(A_i) P(B/A_i)}$$

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**Q4(a)** Suppose that airplane engines operate independently and fail with probability equal to 0.4. Assuming that a plane make a safe flight at least one half of its engine run, determine whether a 4 engin plane is safer than the 2 engin plane justify your answer.

**Solution:-** For the 2 engin plane we want the probability that 1 or more engines are operating successfully

Solved by Labeeb Ahmad

$P(1) + P(2)$ —— (i)

For the 4 engine plane we want the probability that 2 or more engin are successfull

$= P(2) + P(3) + P(4)$ —— (ii)

Probability of faliure = 0.4

Solved by
Labeeb Ahmad

Here $q = 0.4$

$P = 1 - q = 1 - 0.4 = 0.6$

So Take (1)

$$= {}^2C_1 (0.6)^1 (0.4)^{2-1} + {}^2C_2 (0.6)^2 (0.4)^0$$

$$= 0.48 + 0.36$$

$$= 0.84$$

Now take (ii)

$$= {}^4C_2 (0.6)^2 (0.4)^2 + {}^4C_3 (0.6)^3 (0.4)^1 + {}^4C_4 (0.6)^4 (0.4)^0$$

$$= 0.3456 + 0.3456 + 0.1296$$

$$= 0.8208$$

**Solved by
Labeeb Ahmad**

**Q4(b)** Find First three moments about mean for Binomial Distribution.

**Solution:-**

$\mu_1 = E(x) = np$ (See A/2011 pg 29)

$\mu_2 = E(x^2) - [E(x)]^2 = np(1-p)$

(See A/2011 Pg 30)

3rd moment

$E(x^3) = E[(x)(x-1)(x-2) + 3(x)(x-1) + x]$

$= E[x(x-1)(x-2)] + 3E[x(x-1)] + E(x)$ — (i)

$E(x)(x-1)(x-2) = \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x}$

$= \sum_{x=0}^{n} x(x-1)(x-2) \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$

$= \sum_{x=0}^{n} x(x-1)(x-2) \cdot \frac{n(n-1)(n-2)(n-3)!}{x(x-1)(x-2)(x-3)!(n-x)!} p^x q^{n-x}$

$= \sum_{x=0}^{n} n(n-1)(n-2) \cdot \frac{(n-3)!}{(x-3)!(n-x)!} p^x q^{n-x}$

$= n(n-1)(n-2) \cdot p^3 \sum_{x=0}^{n} \frac{(n-3)!}{(x-3)![(n-3)-(x-3)]!} p^{x-3} \cdot q^{n-x}$

$=$

Put $n-3 = m$

And $x-3 = y$

$= n(n-1)(n-2)p^3 \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y}$

Solved by
Labeeb Ahmad

$$E[x(x-1)(x-2)] = n(n-1)(n-2)\,p^3\,(1)$$

Because $\sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y} = 1$

$$= n(n-1)(n-2)\,p^3 \quad \text{—— (ii)}$$

Now

$$E[x(x-1)] = \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \cdot \frac{n!}{x!(n-x)!} \cdot p^x q^{n-x}$$

**Solved by Labeeb Ahmad**

$$= \sum_{x=0}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)(n-x)!} \cdot p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n(n-1)\,(n-2)!}{(x-2)[(n-2)-(x-2)]!} \; p^2 \cdot p^{x-2} \cdot q^{[(n-2)-(x-2)]}$$

put $n-2 = m$ & $x-2 = y$

$$= n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!(m-y)!}\, p^y \cdot q^{m-y}$$

$$= n(n-1)p^2 \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y}$$

$$E[x(x-1)] = n(n-1)p^2 \quad \text{—— (iii)}$$

Put the value of (ii) & (iii) in (i)

$$E(x^3) = n(n-1)(n-2)p^3 + 3[(n)(n-1)p^2] + np$$

$$\mu_3 = E(x^3) - 3E(x)E(x^2) + 2[E(x)]^3$$

$$= (n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np) - 3(np)(n)(n-1)p^2 + 2(np)^3$$

$$\mu_3 = np[1 - 3p + 2p^2] = np(1-p)(1-2p)$$

$$\mu_3 = npq(1-2p)$$

**Q5(a)** If $\lambda, \mu$ are the deviation of random variable X and Y from their respective means.

Then show that

(i) $r = 1 - \frac{1}{2N}\sum\left[\frac{\lambda_1}{\sigma_x} - \frac{\mu_1}{\sigma_y}\right]^2$

(ii) $r = -1 + \frac{1}{2N}\sum\left[\frac{\lambda_1}{\sigma_x} + \frac{\mu_1}{\sigma_y}\right]^2$

And deduce that $-1 \leq r \leq 1$

**Solution:-** As $\lambda_1 = X - \bar{X}$

and $\mu_1 = Y - \bar{Y}$

Consider (i)

$$r = 1 - \frac{1}{2N}\sum\left[\frac{\lambda_1}{\sigma_x} - \frac{\mu_1}{\sigma_y}\right]^2$$

Take R.H.S

$$1 - \frac{1}{2N}\sum\left[\frac{\lambda_1}{\sigma_x} - \frac{\mu_1}{\sigma_y}\right]^2$$

$$\Rightarrow 1 - \frac{1}{2N}\sum\left[\frac{\lambda_1^2}{\sigma_x^2} + \frac{\mu_1^2}{\sigma_y^2} - \frac{2\lambda_1\mu_1}{\sigma_x\sigma_y}\right]$$

As $\lambda_1 = x - \bar{x}$

and $\mu_1 = y - \bar{y}$

Solved by
Labeeb Ahmad

$$\Rightarrow 1-\frac{1}{2N}\sum\left[\frac{(x-\bar{x})^2}{\sigma_x^2}+\frac{(y-\bar{y})^2}{\sigma_y^2}-\frac{2(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y}\right]$$

$$\Rightarrow 1-\frac{1}{2}\left[\frac{1}{\sigma_x^2}\cdot\frac{\sum(x-\bar{x})^2}{N}+\frac{1}{\sigma_y^2}\cdot\frac{\sum(y-\bar{y})^2}{N}-\frac{2}{\sigma_x\sigma_y}\frac{\sum(x-\bar{x})(y-\bar{y})}{N}\right]$$

$$\Rightarrow 1-\frac{1}{2}\left[\frac{1}{\sigma_x^2}\cdot\sigma_x^2+\frac{1}{\sigma_y^2}\cdot\sigma_y^2-2\frac{Cov(x,y)}{\sigma_x\sigma_y}\right]$$

$$\Rightarrow 1-\frac{1}{2}\left[1+1-2r\right]$$

$$\Rightarrow 1-\frac{1}{2}\left[2-2r\right]$$

$$\Rightarrow 1-1+r = r = L.H.S$$

Hence $r = 1-\frac{1}{2N}\sum\left[\frac{\lambda_1}{\sigma_x}+\frac{\mu_1}{\sigma_y}\right]^2$

**Now Consider (ii)** Take R.H.S

$$-1+\frac{1}{2N}\sum\left[\frac{\lambda_1}{\sigma_x}+\frac{\mu_1}{\sigma_y}\right]^2$$

$$\Rightarrow -1+\frac{1}{2N}\sum\left[\frac{\lambda_1^2}{\sigma_x^2}+\frac{\mu_1^2}{\sigma_y^2}+\frac{2\lambda_1\mu_1}{\sigma_x\sigma_y}\right]$$

As $\lambda_1 = x-\bar{x}$

$\mu_1 = y-\bar{y}$

Solved by
Labeeb Ahmad

$$\Rightarrow -1 + \frac{1}{2N} \sum \left[ \frac{(X-\bar{X})^2}{\sigma_x^2} + \frac{(Y-\bar{Y})^2}{\sigma_y^2} + 2 \cdot \frac{(X-\bar{X})(Y-\bar{Y})}{\sigma_x \sigma_y} \right]$$

$$\Rightarrow -1 + \frac{1}{2} \left[ \frac{1}{\sigma_x^2} \cdot \frac{\sum (X-\bar{X})^2}{N} + \frac{1}{\sigma_y^2} \cdot \frac{\sum (Y-\bar{Y})}{N} + \frac{2}{\sigma_x \sigma_y} \cdot \frac{\sum (X-\bar{X})(Y-\bar{Y})}{N} \right]$$

$$\Rightarrow -1 + \frac{1}{2} \left[ \frac{1}{\sigma_x^2} \cdot \sigma_x^2 + \frac{1}{\sigma_y^2} \cdot \sigma_y^2 + \frac{2}{\sigma_x \sigma_y} Cov(X,Y) \right]$$

$$\Rightarrow -1 + \frac{1}{2} \left[ 1 + 1 + 2 \cdot \frac{Cov(X,Y)}{\sigma_x \sigma_y} \right]$$

$$\Rightarrow -1 + \frac{1}{2} \left[ 2 + 2r \right]$$

**Solved by Labeeb Ahmad**

$= -1 + 1 + r \quad = r = L.H.S$

Hence

$$r = -1 + \frac{1}{2N} \sum \left[ \frac{\lambda_1}{\sigma_x} + \frac{u_1}{\sigma_y} \right]^2$$

**Now** From (i) $r = 1 +$ Positive

$\Rightarrow r \leq 1$

From (ii) $r = -1 +$ Positive

$\Rightarrow r \geq -1$

Thus $-1 \leq r \leq 1$

**Q5(b)** If $X_r$ and $X_s$ are the $r$th and $s$th random variable of random sample of size $n$ drawn from the finite population $\{c_1, c_2 --- c_N\}$ Then $Cov(X_r, X_s) = \frac{-\sigma^2}{N-1}$

**Proof** :- Finite Population

$$\{c_1, c_2, c_3 --- c_N\}$$

For without replacment

$$f(c_1) = \frac{1}{N}$$

$$f(c_1, c_2) = \frac{1}{N} \cdot \frac{1}{N-1}$$

**Solved by Labeeh Ahmad**

Similarly

$$f(c_1, c_2 \text{ ---- } c_N) = \frac{1}{N(N-1) \text{ ---- } [N-(n-1)]}$$

Now

$$Cov(X_r, X_s) = \sum_{i=1}^{N} \sum_{j=1}^{N} (c_i - \mu)(c_j - \mu) \cdot \frac{1}{N(N-1)}$$

$$Cov(X_r, X_s) = \frac{1}{N(N-1)} \cdot \sum_{i=1}^{N} (c_i - \mu) \sum_{j=1}^{N} (c_j - \mu) \quad \text{---} (1)$$

where $i \neq j$

Consider

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} (c_j - \mu) = \sum_{j=1}^{N} (c_j - \mu) - (c_i - \mu)$$

$$\sum_{j=1}^{N}(c_j-\mu)=(c_1-\mu)+(c_2-\mu)+\text{---}(c_N-\mu)-(c_i-\mu)$$
$$=\left[(c_1+c_2+c_3\text{---}c_N)-N\mu\right]-(c_i-\mu)$$
$$=\left(\sum_{i=1}^{N}c_i-N\mu\right)-(c_i-\mu)$$

As we know that

$$\mu=\frac{\sum c_i}{N}$$

**Solved by**
**Labeeb Ahmad**

or $\sum c_i = \mu N$

Also

$$\sigma^2=\sum_{i=1}^{N}\frac{(c_i-\mu)^2}{N} \quad \text{——} (2)$$

Thus $\sum_{j=1}^{N}(c_j-\mu) = (N\mu - N\mu) - (c_i-\mu)$

$$= -(c_i-\mu)$$

So (1) becomes

$$Cov(x_r,x_s)=\frac{1}{N(N-1)}\cdot\sum_{i=1}^{N}(c_i-\mu)\left[-(c_i-\mu)\right]$$

$$=\frac{-1}{N-1}\sum_{i=1}^{N}\frac{(c_i-\mu)^2}{N}$$

Now from (2)

$$Cov(x_r,x_s)=\frac{-\sigma^2}{N-1}$$

Hence proof

**Q6(a)** verify that the chi-square ($\chi^2$) distribution has the following density function.

$$f(\chi^2) = \frac{1}{2^{n/2}\,\Gamma(n/2)} (\chi^2)^{\frac{n}{2}-1} e^{-\chi^2/2}, \quad 0<\chi^2<\infty$$

**Solution:-**

Let the random variables $x_i$ ($i = 1, 2, 3 - - - n$) be normally distributed with mean $\mu_i$ ($i = 1, 2, 3 - - - n$) and variance $\sigma_i^2$. So

$$\chi^2 = U = \sum_{i=1}^{n} \left[ \frac{x_i - \mu_i}{\sigma_i} \right]^2$$

**Solved by Labeeb Ahmad**

Let $z_i = \frac{x_i - \mu_i}{\sigma_i}$, $i = 1, 2, 3 - - - n$ be normally and independently Distributed variable with zero mean and unit variance. Then a random variable expressed by the quantity $U = \chi^2 = \sum_{i=1}^{n} z_i^2$

To obtain the distribution of $\chi^2 = \sum_{i=1}^{n} z_i^2$, we use the moment generating function technique.

The m-g-f of chi-square ($\chi^2$) is

By Def

$$M_o(t) = E(e^{t\chi^2})$$

$$= E[e^{t z_i^2}]$$

$$= E[e^{t(z_1^2+z_2^2+z_3^2+ --- z_n^2)}]$$

$$= E[e^{tz_1^2+tz_2^2+tz_3^2+ --- tz_n^2}]$$

$$= E[\prod_{i=1}^{n} e^{tz_i^2}]$$

$$= \prod_{i=1}^{n} E(e^{tz_i^2}) \quad \text{--- (1)}$$ as all $z_i$ are Independent

Now Take

$$E(e^{tz_i^2}) = \int_{-\infty}^{\infty} e^{tz_i^2} f(z_i) \, dz$$

$$= \int_{-\infty}^{\infty} e^{tz^2} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz$$

Solved by Labeeb Ahmad

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(tz^2 - z^2/2)} \, dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{z^2}{2}[1-2t]} \, dz \quad \text{--- (A)}$$

Put $\theta = \frac{z^2}{2}(1-2t)$ --- (2)

$$d\theta = \cancel{2}z \cdot dz \; \frac{(1-2t)}{\cancel{2}}$$

$$\frac{d\theta}{z(1-2t)} = dz \quad \text{—— (3)}$$

From (2) $\longrightarrow$ $z = \sqrt{\dfrac{2\theta}{1-2t}}$

Put in (3)

$$dz = \frac{d\theta}{\dfrac{\sqrt{2\theta}}{\sqrt{1-2t}}(1-2t)}$$

$$dz = \frac{\sqrt{1-2t}\;d\theta}{\sqrt{2\theta}\;(1-2t)}$$

**Solved by**
**Labeeb Ahmad**

$$dz = \frac{d\theta}{\sqrt{2\theta}\sqrt{1-2t}}$$

And $\theta \longrightarrow \infty$ when $z \longrightarrow \infty$

$\theta \longrightarrow 0$ when $z \longrightarrow 0$

Now Equation (A) becomes

$$E\left[e^{tz_i^2}\right] = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\theta} \cdot \frac{d\theta}{\sqrt{2\theta}\sqrt{1-2t}}$$

$$= \frac{2}{\sqrt{2}\sqrt{\pi}} \cdot \frac{1}{\sqrt{2}\,\sqrt{1-2t}} \int_0^\infty e^{-\theta}\, \theta^{-1/2} . d\theta$$

$$= \frac{2}{2\sqrt{\pi}\,\sqrt{1-2t}} \int_0^\infty e^{-\theta}\, \theta^{\frac{1}{2}-1} . d\theta$$

$$= \frac{1}{\sqrt{\pi}\,\sqrt{1-2t}} \Gamma(1/2)$$

$$= \frac{1}{\sqrt{\pi}\,\sqrt{1-2t}} \sqrt{\pi}$$

$$= (1-2t)^{-1/2}$$

**Solved by Labeeb Ahmad**

put in (1)

$$M_0(t) = \prod_{i=1}^{n} \left[ (1-2t)^{-1/2} \right]$$

$$= (1-2t)^{-1/2} \cdot (1-2t)^{-1/2} - \text{—} (1-2t)^{-1/2} \text{ (n times)}$$

$$= (1-2t)^{-n/2} \quad \text{—— (4)}$$

We know that the m.g.f of Gamma (Person Type III) distribution

$$f(x) = \frac{1}{\alpha^m \Gamma(m)} x^{m-1} e^{-x/\alpha} \text{ is}$$

is $M_0(t) = (1-\alpha t)^{-m}$

From (4) $\alpha = 2$ and $m = n/2$

Then $f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} (x)^{\frac{n}{2}-1} e^{-x/2}$

**Solved by**
**Labeeb Ahmad**

$x$ by $\chi^2$

$$f(\chi^2) = \frac{1}{2^{n/2}\Gamma(n/2)} \cdot (\chi^2)^{\frac{n}{2}-1} \cdot e^{-\chi^2/2}$$

where $0 < \chi^2 < \infty$

## Q6 (b) Four pennies were tossed 1000 times and the number of heads were observed as given

| No. of heads | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequencies | 38 | 149 | 352 | 292 | 169 |

Test the Hypothesis whether a Binomial distribution gives the satisfactory fit the data.

## Solution :- we have

Solved by Labeeb Ahmad

$P$(x heads in a toss of 4 Pennies) $= P(x)$

$P(x) = \binom{4}{x} p^x q^{4-x}$

where $p$ and $q$ are respective probabilities of head and tail on the single toss of Penny

As $\mu = np = 4p$

For the actual (or observed) frequency distribution the mean number of head is

$$\frac{\sum f(x)}{\sum f} = \frac{(38)(0)+(149)(1)+(352)(2)+(292)(3)+(169)(4)}{1000}$$

$$= \frac{2405}{1000} = 2.405$$

Equating the theoretical and actual means

$$5P = 2.405$$

**Solved by Labeeb Ahmad**

$$P = \frac{(2.405)}{5} = 0.481$$

$$q = 1-P = 1-0.481$$

$$= 0.519$$

Thus the fitted binomal distribution is given by

$$P(x) = \binom{4}{x}(0.481)^{x}(0.519)^{4-x}$$

---

A/2015, A/2011, A/2003

**Q7(a)** If the joint probability density of $X_1$ and $X_2$ is given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & x_1, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability of $Y = \frac{X_1}{X_1 + X_2}$

**Solution :-** Here only one variable given "Y". But As we know that To apply Jacobian transformation.

There must be number of variables are same.

There for here we let another variable "$Y_1$"

So after applying Jacobian Transformation we get new transformed function like $f(Y, Y_1)$

Finally we integrate $f(Y, Y_1)$ w.r.t "$Y_1$"

Then we get result.

Let $Y_1 = X_1 + X_2$

**Note:** we also let $Y_1 = X_2$ answer will be same

Given $Y = \frac{X_1}{X_1 + X_2}$

$$Y = \frac{X_1}{Y_1} \quad \text{or} \quad X_1 = Y Y_1$$

Solved by
Labeeb Ahmad

Put $X_1 = Y \cdot Y_1$ in (i) $\Rightarrow Y_1 = Y \cdot Y_1 + X_2$

$$Y_1 - Y Y_1 = X_2$$

$$Y_1(1 - Y) = X_2$$

Now we chang $X_1, X_2$ into $Y, Y_1$. There for we use Jacobian Transformation.

$$J = \frac{\partial(x_1, x_2)}{\partial(Y\, Y_1)} = \begin{vmatrix} \frac{\partial x_1}{\partial Y} & \frac{\partial x_1}{\partial Y_1} \\ \frac{\partial x_2}{\partial Y} & \frac{\partial x_2}{\partial Y_1} \end{vmatrix}$$

$$J = \begin{vmatrix} Y_1 & Y \\ -Y_1 & 1-Y \end{vmatrix}$$

$J = Y_1(1-Y) - (Y)(-Y_1)$

$J = Y_1 - YY_1 + YY_1$

$J = Y_1$

$f(Y, Y_1) = f(x_1, x_2)\ |J|$

$f(Y, Y_1) = e^{-Y_1} \cdot Y_1 \qquad Y_1 > 0$

Now we integrate w.r.t $Y_1$

$$f(Y) = \int_0^\infty e^{-Y_1}\, Y_1^{2-1}\, dY_1$$

$$= \Gamma(2)$$

$$= 1!$$

$f(Y) = 1$ Hence prove

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# Q7(b) Find $\mu_{2r}$ for t- distribution

## For $\mu_{2r}$

Because all odd order moments of "t" distribution. There for $\mu'_{2r} = \mu_{2r}$

$$\mu_{2r} = E[t^{2r}] = \int_{-\infty}^{\infty} t^{2r} \cdot f(t) \cdot dt$$

$$= \int_{-\infty}^{\infty} t^{2r} \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} \left[1+\frac{t^2}{n}\right]^{-\frac{1}{2}[n+1]} dt$$

$$= \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} \, 2\int_{0}^{\infty} t^{2r} \left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)} dt$$

$\downarrow$ $I_1$

Take $I_1$ and put

$$\left(1+\frac{t^2}{n}\right)^{-1} = w$$

OR

$$1+\frac{t^2}{n} = w^{-1}$$

Solved by
Labeeb Ahmad

$$t^2 = n[w^{-1} - 1]$$

Differentiate both side

$$2t \cdot dt = n[-w^{-2} \cdot dw]$$

$$2\sqrt{n(w^{-1}-1)}\, dt = -nw^{-2} \cdot dw$$

$\begin{cases} w \to 0 & \text{when } t \to \infty \\ w \to 1 & \text{when } t \to 0 \end{cases}$

$$I_1 = \int_1^0 \left(\sqrt{n(w^{-1}-1)}\right)^{2r} w^{\frac{n+1}{2}} \frac{-nw^{-2}}{2\sqrt{n(w^{-1}-1)}} \cdot dw$$

$$= \frac{-1}{2}\int_1^0 \left(\sqrt{n(w^{-1}-1)}\right)^{2r-1} \cdot n w^{\frac{n+1}{2}-2} \cdot dw$$

$$= \frac{1}{2}\int_0^1 \left[\left(n\left[\frac{1}{w}-1\right]\right)^{\frac{1}{2}}\right]^{2r-1} n w^{\frac{n}{2}-\frac{3}{2}}\, dw$$

$$= \frac{n}{2}\int_0^1 \left[n\frac{(1-w)}{w}\right]^{r-\frac{1}{2}} w^{\frac{n-3}{2}} \cdot dw$$

$$= \frac{n}{2}\int_0^1 n^{r-\frac{1}{2}} \frac{(1-w)^{r-\frac{1}{2}}}{w^{r-\frac{1}{2}}} \cdot w^{\frac{n}{2}-\frac{3}{2}} \cdot dw$$

$$= \frac{n^{r-\frac{1}{2}+1}}{2}\int_0^1 (1-w)^{r-\frac{1}{2}}\, w^{\frac{n}{2}-\frac{3}{2}-r+\frac{1}{2}} \cdot dw$$

$$= \frac{n^{r+\frac{1}{2}}}{2}\int_0^1 w^{\frac{n}{2}-r-1}(1-w)^{r+\frac{1}{2}} \cdot dw$$

$$I_1 = \frac{n^{r+\frac{1}{2}}}{2}\, \beta\left(\frac{n}{2}-r,\ r+\frac{1}{2}\right)$$

**Solved by**
**Labeeb Ahmad**

Putting the value of $I_1$

$$\mu_{2r} = \frac{\cancel{2}}{\sqrt{n}\,\beta\left(\frac{n}{2}, \frac{1}{2}\right)} \cdot \frac{n^{r+\frac{1}{2}}}{\cancel{2}} \cdot \beta\left(\frac{n}{2} - r, r + \frac{1}{2}\right)$$

$$\mu_{2r} = \frac{n^{r}\,\beta\left(\frac{n}{2} - r, r + \frac{1}{2}\right)}{\beta\left(\frac{n}{2}, \frac{1}{2}\right)} \quad \text{——— (A)}$$

As we know that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

using the above property in (A)

$$\mu_{2r} = n^{r}\,\frac{\Gamma\left(\frac{n}{2} - r\right)\Gamma\left(r + \frac{1}{2}\right) \Big/ \cancel{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right) \Big/ \cancel{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}}$$

Hence

$$\mu_{2r} = \frac{n^{r}\,\Gamma\left(\frac{n}{2} - r\right)\Gamma\left(r + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

**Solved by**
**Labeeb Ahmad**

# Mean and Variance of 't' Distribution

$$\text{Mean} = E(t) = \int_{-\infty}^{\infty} t \cdot f(t) \cdot dt$$

As we know all odd order moment of t-distribution is zero there for **Mean = 0**

$$\text{Variance} = E(t^2) - [E(t)]^2$$

$$\mu_2 = E(t^2) \qquad (\because E(t) = 0)$$

As we know

$$\mu_{2r} = \frac{n^r \Gamma(\frac{n}{2} - r)\, \Gamma(r + 1/2)}{\Gamma(\frac{n}{2})\, \Gamma(1/2)}$$

Solved by Labeeb Ahmad

Put $r = 1$

$$\mu_2 = \frac{n\, \Gamma(\frac{n}{2} - 1)\, \Gamma(1 + 1/2)}{\Gamma(n/2)\, \Gamma(1/2)}$$

$$= \frac{n\, \Gamma(\frac{n}{2} - 1) \frac{1}{2} \Gamma(1/2)}{\Gamma(n/2)\, \Gamma(1/2)} = \frac{n}{2} \frac{\Gamma(\frac{n}{2} - 1)}{\Gamma(n/2)}$$

$$\mu_2 = \frac{n}{2} \frac{(\frac{n-2}{2}) \Gamma(\frac{n}{2} - 1)}{(\frac{n-2}{2}) \Gamma(n/2)} = \frac{n\, \Gamma(n/2)}{(n-2)\, \Gamma(n/2)}$$

$$\mu_2 = \frac{n}{n-2} \quad \text{for } (n > 2)$$

A/2015, A/2014, A/2013

## Q8(a)

Show that Co-efficient of Corelation is independent of change of Origin and scale. i-e $r_{xy} = r_{uv}$

**Solution :-** As $r_{xy} = \frac{\sum (X-\bar{X})(Y-\bar{Y})}{\sqrt{\sum (X-\bar{X})^2 \sum (Y-\bar{Y})^2}}$

let $u$ and $v$ be the two new variables defined by

$$u = \frac{X-a}{h} \quad ; \quad v = \frac{Y-b}{k}$$

So $X = uh + a \quad ; \quad Y = vk + b$

where "a" and "b" are the new Origins and "h" and "k" are the units of measurement.

Solved by
Labeeb Ahmad

putting the value of "X" and Y in $r_{xy}$

$$r_{xy} = \frac{\sum [uh + a - a - \bar{u}h][vk + b - \bar{v}k - b]}{\sqrt{\sum (uh + a - \bar{u}h - a)^2 \sum (vk + b - \bar{v}k - b)^2}}$$

$$r_{xy} = \frac{\sum [h(u-\bar{u})][k(v-\bar{v})]}{\sqrt{\sum [h(u-\bar{u})]^2 \sum [k(v-\bar{v})]^2}}$$

$$r_{xy} = \frac{hk\sum (U-\bar{U})(V-\bar{V})}{\sqrt{\sum h^2 (U-\bar{U})^2 \sum k^2 (V-\bar{V})^2}}$$

$$r_{xy} = \frac{\cancel{hk}\sum (U-\bar{U})(V-\bar{V})}{\cancel{hk}\sqrt{\sum (U-\bar{U})^2 \sum (V-\bar{V})^2}}$$

$$r_{xy} = \frac{\sum (U-\bar{U})(V-\bar{V})}{\sqrt{\sum (U-\bar{U})^2 \sum (V-\bar{V})^2}}$$

$r_{xy} = r_{uv}$ Hence prove

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## Q8(b)

Ten individuals are chosen at random from a normal population and the heights are found to be in inches 63, 63, 66, 67, 68, 69, 70, 70, 71 and 71. In the heights of these data, discuss the suggestion that mean height in the population is 66 inches. Use 5% significance level for testing purposes.

Solved by
Labeeb Ahmad

## Solution :-

We set $H_0 : \mu = 66''$ against
$H_1 : \mu \neq 66''$

| $x$ | $x - \bar{x}$ | $(x - \bar{x})^2$ |
|---|---|---|
| 63 | -4.8 | 23.04 |
| 63 | -4.8 | 23.04 |
| 66 | -1.8 | 3.24 |
| 67 | -0.8 | 0.64 |
| 68 | 0.2 | 0.04 |
| 69 | 1.2 | 1.44 |
| 70 | 2.2 | 4.84 |
| 70 | 2.2 | 4.84 |
| 71 | 3.2 | 10.24 |
| 71 | 3.2 | 10.24 |

$$\bar{X} = \frac{678}{10} = 67.8$$

$$\therefore \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{3.011}{\sqrt{10}} = 0.95$$

We compute

$$t = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{67.8 - 66}{0.95}$$

$$= 1.89$$

Two tailed table value of t with 9 degree of freedom. $\sigma$ f at 5% level is found to be 2.26

As computed values as t < tabulated "t". So computed t does not lie in critical region. we accept $H_0$ and conclude that the experiment provides no ground for doubt that the mean hight is 66".

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**Q9(a)** If $F$ follows $F(\nu_1, \nu_2)$ then $y = \left[1 + \frac{\nu_1}{\nu_2} F\right]^{-1}$ follows $\beta\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)$

where $\nu_1, \nu_2$ are the degree of freedom for F-distribution.

## Solution:-

As
$$F(\nu_1, \nu_2) = \frac{(\nu_1/\nu_2)^{\frac{\nu_1}{2}}}{\beta\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} F^{\frac{\nu_1}{2}-1} \frac{1}{\left[1 + \frac{\nu_1 F}{\nu_2}\right]^{\frac{\nu_1+\nu_2}{2}}}$$

And

$$dF = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{\beta\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} F^{\frac{\nu_1}{2}-1} \frac{1}{\left[1 + \frac{\nu_1}{\nu_2} F\right]^{\frac{\nu_1+\nu_2}{2}}} \, dF \quad \text{--- (1)}$$

As
$$y = \left[1 + \frac{\nu_1}{\nu_2} F\right]^{-1}$$

$$y^{-1} = \left[1 + \frac{\nu_1 F}{\nu_2}\right] \quad \text{--- (2)}$$

$$y^{-1} - 1 = \frac{\nu_1}{\nu_2} F$$

Solved by
Labeeb Ahmad

$$\left(\frac{\nu_1}{\nu_2}\right)^{-1} \left[y^{-1} - 1\right] = F \quad \text{--- (3)}$$

Diff both side

$$\left(\frac{v_1}{v_2}\right)^{-1}\left[-y^{-2}\,.\,dy\right] = dF \quad \text{——} \quad (4)$$

Putting the value of equations (2), (3) & (4) in (1)

$$dF = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \left[\left(\frac{v_1}{v_2}\right)^{-1}\left\{\frac{1}{y}-1\right\}\right]^{\frac{v_1}{2}-1} . \frac{1}{\left(y^{-1}\right)^{\frac{v_1}{2}+\frac{v_2}{2}}} . \left(\frac{v_1}{v_2}\right)^{-1}\left[-y^{-2}\right] dy$$

$$dF = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} \left(\frac{v_1}{v_2}\right)^{-\frac{v_1}{2}+1} \left(\frac{1-y}{y}\right)^{\frac{v_1}{2}-1}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} . y^{\frac{v_1}{2}+\frac{v_2}{2}} . \left(\frac{v_1}{v_2}\right)^{-1}\left(-y^{-2}\right) . dy$$

$$dF = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}-\frac{v_1}{2}+1-1} (1-y)^{\frac{v_1}{2}-1} \left(y^{-1}\right)^{\frac{v_1}{2}-1} . y^{\frac{v_1}{2}+\frac{v_2}{2}} \left(-y^{-2}\right) dy}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)}$$

$$dF = \frac{(1-y)^{\frac{v_1}{2}-1} \, -y^{-\frac{v_1}{2}+1+\frac{v_1}{2}+\frac{v_2}{2}-2} \, dy}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)}$$

Solved by
Labeeb Ahmad

$$= \frac{-1}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} (1-y)^{\frac{v_1}{2}-1} . y^{\frac{v_2}{2}-1} . dy$$

This follows Beta (Ist kind) distribution.

**Q9(b)** Find the moment generating function for Gamma distribution and use it to find Mean and Variance.

## Solution :-

By Definition

$M_o(t) = E(e^{tx})$

$= \int_0^\infty e^{tx} \frac{1}{\Gamma(n)} \cdot x^{n-1} e^{-x} \cdot dx$

$= \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} e^{tx-x} \cdot dx$

$= \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-x(1-t)} \cdot dx$

**Solved by Labeeb Ahmad**

Put $x(1-t) = \theta$

$(1-t)dx = d\theta$

$\theta \rightarrow 0$ when $x \rightarrow 0$

$\theta \rightarrow \infty$ when $x \rightarrow \infty$

$M_o(t) = \frac{1}{\Gamma(n)} \int_0^\infty \left[\frac{\theta}{1-t}\right]^{n-1} e^{-\theta} \cdot \frac{d\theta}{(1-t)}$

$$M_0(t) = \frac{1}{\Gamma(n)} (1-t)^{1-n-1} \int_0^\infty \theta^{n-1} e^{-\theta} \cdot d\theta$$

$$= \frac{(1-t)^{-n}}{\Gamma(n)} \cdot \Gamma(n)$$

$$M_0(t) = (1-t)^{-n} \quad ; |t| < 1$$

**Mean** $= \lim_{t \to 0} \left[ \frac{d}{dt} M_0(t) \right]$

$$\mu_1 = \left[ \frac{d}{dt} (1-t)^{-n} \right]_{t \to 0}$$

$$= \left[ -n (1-t)^{-n-1} (-1) \right|_{t \to 0}$$

$$= +n (1)^{-n-1}$$

$$= n$$

Solved by
Labeeb Ahmad

Now

$$\mu_2' = \lim_{t \to 0} \left[ \frac{d^2}{dt^2} M_0(t) \right]$$

$$= \frac{d}{dt} \left[ n (1-t)^{-n-1} \right|_{t \to 0}$$

$$= n(-n-1)(1-t)^{-n-2}(-1) \Big|_{t \to 0}$$

$$\mu_2' = n(n+1)$$

**Variance** $= \mu_2' - (\mu_1')^2 = n^2 + n - n^2$

$$= n$$

# PAPER-A/2014

Solved by
Labeeb Ahmad

A|2014

## Q1(a)

The contents of Urn I, II and III are as follows

| | | White | Black | Red |
|---|---|---|---|---|
| Urn I | A | 1 | 1 | 1 |
| Urn II | B | 2 | 1 | 1 |
| Urn III | C | 4 | 5 | 3 |

One urn is chosen at random and two balls are drawn they happen to be white and red. What is the probability that they came from Urn II & III?

## Solution :-

$P(A) = P(B) = P(C) = \frac{1}{3}$

Let W denote the event that the ball drawn is white, then

$P(W/A) = \frac{1}{3}$ , $P(W/B) = 2/4$

and $P(W/C) = \frac{4}{12} = \frac{1}{3}$

Solved by
Labeeb Ahmad

Let R be the event that the ball drawn is Red.

$P(R/A) = \frac{1}{3}$ , $P(\frac{R}{B}) = \frac{1}{4}$ , $P(R/C) = \frac{3}{12} = \frac{1}{4}$

Now By Baye's Theorem, we have

$$P(\text{Urn II}: B/W) = \frac{P(W)\ P(W/B)}{P(A)\ P(W/A) + P(B) P(W/B) + P(C) \cdot P(W/C)}$$

$$= \frac{\frac{1}{3} \cdot 1/2}{\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3}}$$

$$= \frac{1/6}{\frac{1}{9} + \frac{1}{6} + \frac{1}{9}} = 0.428$$

And

$$P(\text{Urn III}\ C/R) = \frac{P(R)\ P(R/C)}{P(A)\ P(R/A) + P(B) P(R/B) + P(C)\ P(R/C)}$$

**Solved by Labeeb Ahmad**

$$= \frac{\frac{1}{3} \cdot \frac{1}{4}}{\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}}$$

$$= \frac{1/12}{1/9 + 1/12 + 1/12}$$

$$= 0.298$$

**Q1(b)** Ten digits 1, 2, --- n (n < 10) are written in a random order such that n! arragement are all equally likely to give a number of digits. If n = 7 Then show that the number divisible by 2 is $\frac{2 \times 6!}{7!}$ and the Probability that the number divisible by 4 is $\frac{10 \times 5!}{7!}$

## Solution :-

As $n = 7$

Then $S = \{1, 2, 3, 4, 5, 6, 7\}$

$n(S) = 7$

Let A be the event that the integer is divisible by 2.

$A = \{2, 4, 6\}$

i.e $n(A) = 3$

$P(A) = \frac{n(A)}{n(S)} = \frac{3}{7}$

**Solved by**
**Labeeb Ahmad**

multiply and divide by 6!

$P(A) = \frac{3}{7} \times \frac{6!}{6!} = \frac{3 \times 6!}{7!}$

And let B be the event that chose an integer divisible by 4

$B = \{4\}$ i-e $n(B) = 1$

$$P(B) = \frac{n(B)}{n(S)} = \frac{1}{7}$$

multiply & divide by 6!

$$P(B) = \frac{1}{7} \times \frac{6!}{6!} = \frac{6 \cdot 5!}{7!}$$

$$= \frac{6 \times 5!}{7!}$$

**Solved by**
**Labeeb Ahmad**

A/2014

**Q2. (a)** A letter is Taken at random from the letter of word "STATISTICS" another word is "ASSISTANT". Find the probability that they are same letter.

**Solution:-** Total letter of word STATISTICS is 10.
Total letter of word ASSISTANT is 9.

$S = 10 + 9 = 19$

| Same word | | STATISTICS | ASSISTANT | Total | Probability |
|---|---|---|---|---|---|
| | T | 3 | 2 | 5 | 5/19 |
| | S | 3 | 3 | 6 | 6/19 |
| | I | 2 | 1 | 3 | 3/19 |
| | A | 1 | 2 | 3 | 3/19 |

$$\frac{5}{19} + \frac{6}{19} + \frac{3}{19} + \frac{3}{19} = 17/19$$

## Q2 (b)

Given the joint density function of random variable x and y as

$$f(x,y) = 3x^2 + 3y^2x \quad 0 \le x \le 1$$
$$0 \le y \le 1$$

Find (i) $f_1(x)$ (ii) $f_2(x)$

(iii) $P\left(\frac{1}{3} \le y \le 2/3 \,/\, \frac{1}{2} \le x \le 3/4\right)$

## Solution :-

Given $f(x,y) = 3x^2 + 3y^2x$

(i) $f_1(x) = g(x) = \int_{-\infty}^{\infty} f(x,y) \cdot dy$

$$= \int_{-\infty}^{0} f(x,y) \cdot dy + \int_{0}^{1} f(x,y) \cdot dy + \int_{1}^{\infty} f(x,y) \cdot dy$$

$$= 0 + \int_{0}^{1} (3x^2 + 3y^2x) \cdot dy + 0$$

Solved by
Labeeb Ahmad

$$= 3x^2 [y|_0^1 + 3x \cdot [\frac{y^3}{3}|_0^1$$

$$= 3x^2(1-0) + x(1-0)$$

$$= 3x^2 + x$$

$$g(x) = \begin{cases} 3x^2 + x & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Solved by
Labeeb Ahmad

**(ii)**

$$f_2(x) = h(y) = \int_{-\infty}^{\infty} f(x,y)\cdot dx$$

$$= \int_0^1 (3x^2 + 3y^2 x)\cdot dx \quad , \quad 0 \le y \le 1$$

$$= 3\left[\frac{x^3}{3}\right|_0^1 + 3y^2\left[\frac{x^2}{2}\right|_0^1 \quad , \quad 0 \le y \le 1$$

$$= (1-0) + \frac{3}{2}y^2(1-0) \quad , \quad 0 \le y \le 1$$

$$= 1 + \frac{3y^2}{2} \quad , \quad 0 \le y \le 1$$

Hence

$$f_2(x) = h(y) = \begin{cases} 1 + \frac{3}{2}y^2 & ; \ 0 \le y \le 1 \\ 0 & ; \ \text{else where} \end{cases}$$

**(iii)**

$$P\left[\frac{1}{3} \le y \le \frac{2}{3} \Big/ \frac{1}{2} \le x \le \frac{3}{4}\right]$$

$$= \int_{1/3}^{2/3}\int_{1/2}^{3/4} (3x^2 + 3y^2 x)\, dx \cdot dy$$

$$\int_{1/2}^{3/4} (3x^2 + x)\cdot dx$$

$$= \int_{1/3}^{2/3}\left[ \frac{3x^3}{3}\Big|_{1/2}^{3/4} + 3y^2\cdot\frac{x^2}{2}\Big|_{1/2}^{3/4}\right] dy$$

$$\frac{3x^3}{3} + \frac{x^2}{2}\Big|_{1/2}^{3/4}$$

**Solved by**
**Labeeb Ahmad**

$$= \frac{\int_{1/3}^{2/3}\left[\left\{\left(\frac{3}{4}\right)^3-\left(\frac{1}{2}\right)^3\right\}+\frac{3}{2}y^2\left\{\left(\frac{3}{4}\right)^2-\left(\frac{1}{2}\right)^2\right\}\right]dy}{\left\{\left(\frac{3}{4}\right)^3-\left(\frac{1}{2}\right)^3\right\}+\frac{1}{2}\left\{\left(\frac{3}{4}\right)^2-\left(\frac{1}{2}\right)^2\right\}}$$

$$= \frac{\int_{1/3}^{2/3}\left[\left(\frac{27}{64}-\frac{1}{8}\right)+\frac{3}{2}y^2\left(\frac{9}{16}-\frac{1}{4}\right)\right]dy}{\left(\frac{27}{64}-\frac{1}{8}\right)+\frac{1}{2}\left(\frac{9}{16}-\frac{1}{4}\right)}$$

$$= \frac{\int_{1/3}^{2/3}\left[\frac{19}{64}+\frac{3}{2}y^2\left(\frac{5}{16}\right)\right]dy}{\left(\frac{19}{64}\right)+\frac{1}{2}\left(\frac{5}{16}\right)}$$

$$= \frac{\frac{19}{64}\,y\Big|_{1/3}^{2/3}+\frac{15}{32}\cdot\frac{y^3}{3}\Big|_{1/3}^{2/3}}{(29/64)}$$

$$= \frac{\frac{19}{64}\left[\frac{2}{3}-\frac{1}{3}\right]+\frac{15}{32}\times\frac{1}{3}\left[\left(\frac{2}{3}\right)^3-\left(\frac{1}{3}\right)^3\right]}{(29/64)}$$

$$\frac{\frac{19}{64}\left(\frac{1}{3}\right)+\frac{5}{32}\left(\frac{8}{27}-\frac{1}{27}\right)}{(29/64)} = \frac{\frac{241}{1728}}{\frac{29}{64}}$$

$$= \frac{241}{783}$$

**Solved by**
**Labeeb Ahmad**

A/2014



# Q3(a)

Let $x$ be a random variable having the probability density

$$f(x) = \begin{cases} \frac{3}{4}(1+x^2) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E(X) = \mu$ and $Var(X) = \sigma^2$ then find $P\{|X-\mu| < \sigma\}$

## Solution :-

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \int_0^1 x \cdot \frac{3}{4}(1+x^2) \cdot dx$$

$$= \frac{3}{4}\int_0^1 (x+x^3) \cdot dx$$

**Solved by Labeeb Ahmad**

$$= \frac{3}{4}\left[\frac{x^2}{2} + \frac{x^4}{4}\right|_0^1$$

$$= \frac{3}{4}\left[\frac{1}{2} + \frac{1}{4} - 0 - 0\right]$$

$$= \frac{3}{4}\left[\frac{1}{2} + \frac{1}{4}\right] = \frac{9}{16} = 0.5625$$

And

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \cdot dx$$

$$E(X^2) = \int_0^1 x^2 \cdot \frac{3}{4}(1+x^2) \cdot dx$$

$$= \frac{3}{4}\int_0^1 (x^2 + x^4) \cdot dx$$

$$= \frac{3}{4}\left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4}\left[\frac{1}{3} + \frac{1}{5} - 0 - 0\right] = \frac{3}{4}\left(\frac{8}{15}\right) = \frac{2}{5} = 0.4$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 0.4 - (0.5625)^2$$

$$= 0.0836, \text{ so.}$$

$$S.D = \sqrt{Var(X)} = \sigma = \sqrt{0.0836} = 0.289$$

Now $P(|X-\mu| < \sigma) = P(-\sigma < X - \mu < \sigma)$

$$= P(\mu - \sigma < X < \mu + \sigma)$$

$$= P(0.5625 - 0.289 < X < 0.5626 + 0.289)$$

Solved by
Labeeb Ahmad

$$= P(0.2735 < X < 0.8515)$$

$$P(0.2735 < X < 0.8515) = \frac{3}{4}\int_{0.2735}^{0.8515} (1+x^2) \cdot dx$$

$$= \frac{3}{4}\left[x + \frac{x^3}{3}\right]_{0.2735}^{0.8515}$$

$$= \frac{3}{4}\left[0.8515 + \frac{(0.8515)^3}{3} - (0.2735) - \frac{(0.2735)^3}{3}\right]$$

$$= \frac{3}{4}(1.05729 - 0.28032) = 0.5827$$

A/2014 , A/2011 , A/2008 Pg = 96

# Q3(b)

Let the random variable follow the binomial distribution with parameter n and p then show that

$$z = \frac{x-np}{\sqrt{npq}}$$

Tends to the standard normal distribution as $n \to \infty$

## Solution :-

Solved by Labeeb Ahmad

$$z = \frac{x-np}{\sqrt{npq}}$$

The m.g.f of z is

$$M(t) = E(e^{tz}) = E\left[e^{t\left(\frac{x-np}{\sqrt{npq}}\right)}\right]$$

$$= E\left[e^{\frac{tx}{\sqrt{npq}} - \frac{tnp}{\sqrt{npq}}}\right]$$

$$= e^{-\frac{tnp}{\sqrt{npq}}} \cdot E\left[e^{tx/\sqrt{npq}}\right]$$

$$= e^{-\frac{tnp}{\sqrt{npq}}}\left[\sum_{x=0}^{n}\binom{n}{x}p^{x}q^{n-x}\left(e^{t/\sqrt{npq}}\right)^{x}\right]$$

Solved by Labeeb Ahmad

$$= e^{-\frac{tnp}{\sqrt{npq}}}\left[\sum_{x=0}^{n}\binom{n}{x}q^{n-x}\left(pe^{t/\sqrt{npq}}\right)^{x}\right]$$

$$M(t) = e^{\frac{-tnp}{\sqrt{npq}}}\left[q + pe^{t/\sqrt{npq}}\right]^n$$

$$= \left[e^{-tp/\sqrt{npq}}\right]^n \left[q + pe^{t/\sqrt{npq}}\right]^n$$

$$= \left[qe^{-tp/\sqrt{npq}} + pe^{\frac{-tp}{\sqrt{npq}} + \frac{t}{\sqrt{npq}}}\right]^n$$

$$= \left[qe^{-tp/\sqrt{npq}} + pe^{\frac{t}{\sqrt{npq}}(-p+1)}\right]^n$$

Solved by Labeeb Ahmad

$$= \left[qe^{-tp/\sqrt{npq}} + pe^{tq/\sqrt{npq}}\right]^n$$

As we know $e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots$

So

$$= \left[q\left\{1 - \frac{tp}{\sqrt{npq}} + \frac{1}{2!}\left(\frac{tp}{\sqrt{npq}}\right)^2 - \frac{1}{3!}\left(\frac{pt}{\sqrt{npq}}\right)^3 + \ldots\right\}\right.$$

$$\left. + p\left\{1 + \frac{tq}{\sqrt{npq}} + \frac{1}{2!}\left(\frac{tq}{\sqrt{npq}}\right)^2 + \frac{1}{3!}\left(\frac{tq}{\sqrt{npq}}\right)^3 + \ldots\right\}\right]^n$$

Solved by
Labeeb Ahmad

$$= \left[q - \frac{tpq}{\sqrt{npq}} + \frac{1}{2!}\frac{t^2p^2q}{npq} - \frac{1}{3!}\frac{p^3t^3q}{npq\sqrt{npq}} + \ldots\right.$$

$$\left. + p + \frac{tpq}{\sqrt{npq}} + \frac{1}{2!}\frac{t^2q^2p}{npq} + \frac{1}{3!}\frac{t^3q^3p}{npq\sqrt{npq}} + \ldots\right]^n$$

$$M(t) = \left[(p+q) + \frac{1}{2!}\frac{t^2p^2q}{npq} + \frac{1}{2!}\frac{t^2q^2p}{npq}\right.$$

$$\left. - \frac{1}{3!}\frac{p^3t^3q}{npq\sqrt{npq}} + \frac{1}{3!}\frac{t^3q^3p}{npq\sqrt{npq}} + \cdots\right]^n$$

$$M(t) = \left[1 + \frac{t^2pq(p+q)}{2!\,(npq)} + \frac{1}{3!}\frac{t^3pq(-p^2+q^2)}{npq\sqrt{npq}} + \cdots\right]^n$$

Taking ln both side

$$\ln M(t) = \ln\left[1 + \frac{t^2}{2!}\cdot\frac{1}{n} + \frac{1}{3!}\frac{t^3(q^2-p^2)}{n\sqrt{npq}} + \cdots\right]^n$$

$$= n\ln\left[1 + \left(\frac{t^2}{2!}\cdot\frac{1}{n} - \frac{1}{3!}\frac{t^3(q^2-p^2)}{n\sqrt{npq}} + \cdots\right)\right]$$

Using $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$

$$= n\left[\left(\frac{t^2}{2!}\cdot\frac{1}{n} + \frac{1}{3!}\frac{t^3(q-p)}{n\sqrt{npq}} + \cdots\right)\right.$$

$$\left. - \frac{1}{2}\left(\frac{t^2}{2!}\cdot\frac{1}{n} - \frac{1}{3!}\frac{t^3(q-p)}{n\sqrt{npq}} + \cdots\right)^2 + \cdots\right]$$

Solved By
Labeeb Ahmad

$$\ln M(t) = \frac{t^2}{2!} + \frac{t^3}{3!}\frac{(q-p)}{\sqrt{npq}} + \cdots$$

Now Taking $\lim_{n\to\infty}$

$$\lim_{n\to\infty} \ln M(t) = \lim_{n\to\infty}\left[\frac{t^2}{2!} + \frac{t^3}{3!}\frac{(q-p)}{\sqrt{npq}} + \cdots\right]$$

$$= \frac{t^2}{2!} + 0 + 0 + 0 \cdots$$

$$\lim_{n\to\infty} \ln M(t) = \frac{t^2}{2!}$$

Solved by
Labeeb Ahmad

OR

$$\lim_{n\to\infty} M(t) = e^{t^2/2!}$$

We know that $e^{t^2/2}$ is the m-g-f of standard normal variable

Hence we find that in the limit z has a standard normal Distribution.

A/2014 $\underline{\underline{100}}$

**Q4(a)** If $x_1, x_2 --- x_n$ are r-v's let $Y_1 = \sum_{i=1}^{n} a_i x_i$ and $Y_2 = \sum_{i=1}^{n} b_i x_i$

Then show that

$$Cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i Var(x_i) + \sum\sum_{i \neq j} (a_i b_j + a_j b_i) Cov(x_i, y_j)$$

## Solution :-

$$Cov(X, Y) = E[\{X - E(X)\}\{Y - E(Y)\}]$$

$$Cov(Y_1, Y_2) = E[(Y_1 - E(Y_1))(Y_2 - E(Y_2))]$$

**Solved by Labeeb Ahmad**

$$= E\left(\left[\sum_{i=1}^{n} a_i x_i - E\left(\sum_{i=1}^{n} a_i x_i\right)\right]\left[\sum_{i=1}^{n} b_i x_i - E\left(\sum_{i=1}^{n} b_i x_i\right)\right]\right)$$

$$= E\left(\left[\sum_{i=1}^{n} a_i (x_i - E(x_i))\right]\left[\sum_{i=1}^{n} b_i (x_i - E(x_i))\right]\right)$$

$$= \sum_{i=1}^{n} a_i b_i E[x_i - E(x_i)]^2 +$$

$$\sum\sum_{i \neq j} E\left([x_i - E(x_i)][x_j - E(x_j)]\right) +$$

$$\sum\sum_{j \neq i} a_j b_i E\left([x_j - E(x_j)][x_i - E(x_i)]\right)$$

$$= \sum_{i=1}^{n} a_i b_j Var(x_i) + \sum\sum_{i \neq j} (a_i b_j + a_j b_i) E\left([x_i - E(x_i)][x_j - E(x_j)]\right)$$

$$= \sum_{i=1}^{n} a_i b_j Var(x_i) + \sum\sum_{i \neq j} (a_i b_j + a_j b_i) Cov(x_i, y_j)$$

A/2014, A/2008 

# Q4(b)

For the Poisson distribution

Prove $\mu_{r+1} = \lambda\left[r\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right]$

Solution :-

$$\mu_r = E(x-\lambda)^r$$ rth moment about mean

By Definition

$$\mu_r = E(x-\lambda)^r = \sum_{x=0}^{\infty} \frac{(x-\lambda)^r e^{-\lambda}\lambda^x}{x!}$$

$$\mu_r = \sum_{x=0}^{\infty} \frac{1}{x!}\left[(x-\lambda)^r(e^{-\lambda}\lambda^x)\right]$$

Diff w r t "$\lambda$"

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{1}{x!}\left[(x-\lambda)^r\{e^{-\lambda}x\lambda^{x-1} + \lambda^x e^{-\lambda}(-1)\}\right.$$

Ist as it — Derivation of 2nd

$$\left. + (e^{-\lambda}\lambda^x)\{r(x-\lambda)^{r-1}(-1)\}\right]$$

2nd as it — Derivation of Ist

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{1}{x!}(x-\lambda)^r e^{-\lambda}\lambda^x\left[\frac{x}{\lambda} - 1\right]$$

$$-r\sum_{x=0}^{\infty} \frac{1}{x!}(x-\lambda)^{r-1} \cdot e^{-\lambda}\lambda^x$$

Solved by
Labeeb Ahmad

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{1}{x!}(x-\lambda)^r e^{-\lambda} \lambda^x \left[\frac{x-\lambda}{\lambda}\right]$$

$$- r\mu_{r-1}$$

$$= \frac{1}{\lambda}\sum_{x=0}^{\infty} \frac{(x-\lambda)^{r+1} e^{-\lambda} \lambda^x}{x!} - r\mu_{r-1}$$

$$\frac{d\mu_r}{d\lambda} = \frac{1}{\lambda}\mu_{r+1} - r\mu_{r-1}$$

multiplying both side by $\lambda$

$$\lambda \frac{d\mu_r}{d\lambda} = \mu_{r+1} - \lambda r \mu_{r-1}$$

$$\mu_{r+1} = \lambda r \mu_{r-1} + \lambda \frac{d\mu_r}{d\lambda}$$

$$\mu_{r+1} = \lambda\left[r\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right]$$

Solved by
Labeeb Ahmad

Solved by
Labeeb Ahmad

**Q5(a)** Obtain the moment generating function of the gamma distribution And obtain the coefficient of skewness and kurtosis.

## Solution :-

By Definition

$M_o(t) = E(e^{tx}) = \int_0^\infty e^{tx} \frac{x^{n-1} e^{-x}}{\Gamma(n)} . dx$

$= \frac{1}{\Gamma(n)} . \int_0^\infty x^{n-1} e^{-(1-t)x} . dx$

put $(1-t)x = \theta$

$(1-t).dx = d\theta$

Solved by Labeeb Ahmad

$\theta \to 0$ when $x \to 0$

$\theta \to \infty$ when $x \to \infty$

$M_o(t) = \frac{1}{\Gamma(n)} \int_0^\infty \left(\frac{\theta}{1-t}\right)^{n-1} e^{-\theta} . \frac{d\theta}{(1-t)}$

$= \frac{1}{\Gamma(n)} . \frac{1}{(1-t)^{n-1+1}} \int_0^\infty \theta^{n-1} e^{-\theta} . d\theta$

$= \frac{(1-t)^{-n} . \Gamma(n)}{\Gamma(n)}$

$M_o(t) = (1-t)^{-n} \quad ; |t|<1$

For c-g-f Taking ln both side

$\ln M_o(t) = \ln (1-t)^{-n}$

$= -n \ln(1-t)$

$= -n\left[-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \cdots\right]$

$K(t) = n\left[t + \frac{t^2}{2!} + 2!\frac{t^3}{3!} + 3!\frac{t^4}{4!} + \cdots\right]$

$K_1 t + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + K_4 \cdot \frac{t^4}{4!} + \cdots K_r\frac{t^r}{r!} + \cdots$

$= n\left[t + \frac{t^2}{2!} + \frac{2t^3}{3!} + \frac{6t^4}{4!} + \cdots\right]$

Compare the co-efficient of like Powre of 't' we get

$K_1 = \mu = n$

$K_2 = \mu_2 = n$ **Solved by**

$K_3 = \mu_3 = 2n$ **Labeeb Ahmad**

$K_4 = \mu_4 - 3\mu_2^2 = 6n \Rightarrow \mu_4 = 6n + 3n^2$

Now Coefficiet of Skewness

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(2n)^2}{(n)^3} = \frac{4n^2}{n^3} = \frac{4}{n}$

And Kurtosis $= \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{6n+3n^2}{n^2} = \frac{6+3n}{n}$

**Q5(b)** Show that the Beta Distribution becomes the Gamma Distribution as $n \to \infty$

## Solution :-

As $d\beta(\alpha,\beta) = \frac{1}{\beta(\alpha,\beta)} . x^{\alpha-1}(1-x)^{\beta-1} .dx$

Let $z = \beta x \Rightarrow dz = \beta dx$

$dx = \frac{1}{\beta} . dz$

$x = \frac{z}{\beta}$

Then $d\beta(\alpha,\beta) = \frac{1}{\beta(\alpha,\beta)} \left(\frac{z}{\beta}\right)^{\alpha-1} \left(1-\frac{z}{\beta}\right)^{\beta-1} \left(\frac{1}{\beta}\right) dz$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} . (z^{\alpha-1}) \left(\frac{1}{\beta}\right)^{\alpha-1} \left(1-\frac{z}{\beta}\right)^{\beta} \left(1-\frac{z}{\beta}\right)^{-1} \left(\frac{1}{\beta}\right) . dz$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \frac{z^{\alpha-1}}{\beta^{\alpha}} \left[\left(1-\frac{z}{\beta}\right)^{-\frac{\beta}{z}}\right]^{-z} \left(1-\frac{z}{\beta}\right)^{-1} dz$

Now Applying $\lim_{\beta \to \infty}$

Solved by
Labeeb Ahmad

$$\lim_{\beta\to\infty} d\,\beta(\alpha,\beta) = z^{\alpha-1} \lim_{\beta\to\infty} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\,\beta^{\alpha}}$$

$$\lim_{\beta\to\infty}\left[\left(1-\frac{z}{\beta}\right)^{-\frac{\beta}{z}}\right]^{-z} \cdot \lim_{\beta\to\infty}\left[1-\frac{z}{\beta}\right]^{-1} dz \quad \text{——— (1)}$$

Taking $\lim_{\beta\to\infty} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)\,\beta^{\alpha}}$

$$= \lim_{\beta\to\infty} \frac{(\alpha+\beta-1)!}{\Gamma(\alpha)\,\Gamma(\beta)\,\beta^{\alpha}}$$

$$= \lim_{\beta\to\infty} \frac{(\alpha+\beta-1)(\alpha+\beta-2)\text{———}(\alpha+\beta-\alpha)\,\Gamma(\alpha+\beta-\alpha)}{\Gamma(\alpha)\,\Gamma(\beta)\,\beta^{\alpha}}$$

سب ٹرمز میں سے $\beta$ Common لیا کیونکہ یہ ٹوٹل $\alpha$ ٹرمز ہیں اس لیے $\beta^{\alpha}$ نکلا

$$= \lim_{\beta\to\infty} \frac{\beta^{\alpha}\left[1+\frac{\alpha-1}{\beta}\right]\left[1+\frac{\alpha-2}{\beta}\right]\text{————}}{\Gamma(\alpha)\,\beta^{\alpha}}$$

$$= \frac{(1)(1)(1)\text{————}(1)}{\Gamma(\alpha)}$$

$$= \frac{1}{\Gamma(\alpha)}$$

Taking $\lim_{\beta\to\infty}\left[\left(1-\dfrac{z}{\beta}\right)^{-\beta/z}\right]^{-z}$

$$e^{-z}$$

**Solved by Labeeb Ahmad**

Taking

$$\lim_{\beta \to \infty} \left[ \left(1 - \frac{z}{\beta}\right)^{-1} \right]$$

$= 1$

putting these in (1)

$$\lim_{\beta \to \infty} \alpha\, \beta(\alpha, \beta) = z^{\alpha-1} \frac{1}{\Gamma(\alpha)} . e^{-z} . dz$$

$$= \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} . dz$$

$z > 0$

Hence Prove

**Solved by**
**Labeeb Ahmad**

**Q6(a)** Obtain the mean, median mode and Mean deviation of Normal distribution with p-d-f

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} ; -\infty < x < \infty$$

## Solution:-

**Mean** $= E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} . dx$

$$\text{Mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \, e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} . dx$$

Solved by
Labeeb Ahmad

Put $\frac{x-\mu}{\sigma} = z$

$\frac{dx}{\sigma} = dz$ or $dx = \sigma dz$

$z \to \infty$ when $x \to \infty$

$z \to -\infty$ when $x \to -\infty$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) \, e^{-z^2/2} \cdot \sigma . dz$$

$$\text{Mean} = \frac{\mu}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-z^2/2}\, dz}_{I_1} + \frac{\sigma}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} z e^{-z^2/2} . dz}_{I_2} \quad \text{(1)}$$

Take

$$I_1 = \int_{-\infty}^{\infty} e^{-z^2/2} . dz = 2\int_{0}^{\infty} e^{-z^2/2} . dz$$

put $\frac{z^2}{2} = t$

$$\frac{2z}{2} . dz = dt \quad \text{or} \quad dz = \frac{dt}{z}$$

$t \to \infty$ when $z \to \infty$

$t \to 0$ when $z \to 0$

$$I_1 = 2\int_0^{\infty} e^{-t} \frac{dt}{z} = 2\int_0^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{2}{\sqrt{2}} \int_0^{\infty} e^{-t}\, t^{-1/2} . dt$$

$$= \sqrt{2} \int_0^{\infty} e^{-t}\, t^{\frac{1}{2}-1} . dt$$

**Solved by Labeeb Ahmad**

$I_1 = \sqrt{2}\ \Gamma(1/2)$

$I_1 = \sqrt{2}\ \sqrt{\pi}$ $\quad\left(\because \Gamma(1/2) = \sqrt{\pi}\right)$

$I_1 = \sqrt{2\pi}$

Now take $I_2$

$$I_2 = \int_{-\infty}^{\infty} z e^{-z^2/2} . dz$$

$$= -\int_{-\infty}^{\infty} e^{-z^2/2} (-z . dz)$$

$$= -\left[ e^{-z^2/2} \right]_{-\infty}^{\infty}$$

$$= -\left[ \frac{1}{e^{z^2/2}} \right]_{-\infty}^{\infty} = -\left[ \frac{1}{e^{\infty}} - \frac{1}{e^{\infty}} \right]$$

$$= -\left[ \frac{1}{\infty} - \frac{1}{\infty} \right] = 0 - 0 = 0$$

$I_2 = 0$

Put the value of $I_1$ & $I_2$ in (1)

$$Mean = \frac{\mu}{\sqrt{2\pi}} \sqrt{2\pi} + \frac{\sigma}{\sqrt{2\pi}} (0)$$

**$Mean = \mu$**

**Solved by**
**Labeeb Ahmad**

///

**Median** :- Now the median $M$ is given by $\int_{-\infty}^{M} f(x)\cdot dx = \frac{1}{2}$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{M} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \cdot dx = \frac{1}{2}$$

put $\frac{x-\mu}{\sigma} = z$

$dx = \sigma . dz$

$z \to -\infty$ when $x \to -\infty$

$z \to \frac{M-\mu}{\sigma}$ when $x \to M$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{1}{\sigma}(M-\mu)} e^{-z^2/2}\, \sigma . dz = \frac{1}{2} \quad \text{—— (i)}$$

But we know from symmetry of standard normal distribution that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2} . dz = 1/2 \quad \text{—— (ii)}$$

Comparing (i) & (ii)

Solved by
Labeeb Ahmad

$\frac{1}{\sigma}(M-\mu) = 0$

$M - \mu = 0$

$$\boxed{M = \mu}$$

**Mode :-** any value of $x$ for which $f'(x) = 0$ and $f''(x) < 0$ is called Mode.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

Diff w.r.t "$x$"

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \cdot -\left(\frac{1}{2}\right) 2\left[\frac{x-\mu}{\sigma^2}\right]$$

Equating $f'(x) = 0$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} (-1)\left[\frac{x-\mu}{\sigma^2}\right] = 0$$

So $\frac{x-\mu}{\sigma^2} = 0$

$x = \mu$

Solved by
Labeeb Ahmad

Now again Diff w.r.t "$x$"

$$f''(x) = \frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left(-\frac{1}{\sigma^2}\right) + \left[-\frac{x-\mu}{\sigma^2}\right]. e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left(-\frac{x-\mu}{\sigma^2}\right)\right]$$

$$f''(x) = \frac{1}{\sigma\sqrt{2\pi}} - e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left(\frac{1}{\sigma^2} - \frac{(x-\mu)^2}{\sigma^4}\right)$$

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left(1 - \left(\frac{x-\mu}{\sigma}\right)^2\right)$$

substituting $x = \mu$ in $f''(x)$

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\mu-\mu}{\sigma}\right]^2}\left(1 - \left(\frac{\mu-\mu}{\sigma}\right)^2\right)$$

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{0} \, (1-0)$$

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}}$$

**Solved by**
**Labeeb Ahmad**

So $f''(x) < 0$

Thus $x = \mu$ is the mode of normal Distribution

And

**M-D** = See A|2016 Q4(a) Pg "**14**"

**Q6(b)** A fair coin is tossed 20 times. Find the Probability that the number of heads occuring between 10 and 14 inclusive using
(i) Binomial Distribution
(ii) Normal Distribution

## Solution :-

Solved by
Labeeb Ahmad

**(i)** Let x denote the number of heads occuring. Then the p.d.f of x is

$$P(X=x) = {}^{20}C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{20-x}$$

$$P(10 \leq X \leq 14) = \sum_{x=10}^{14} {}^{20}C_x \left(\tfrac{1}{2}\right)^{20}$$

$$= \frac{1}{2^{20}}\left[{}^{20}C_{10} + {}^{20}C_{11} + {}^{20}C_{12} + {}^{20}C_{13} + {}^{20}C_{14}\right]$$

$$= \frac{1}{1048576}\left[184756 + 167960 + 125970 + 77520 + 38760\right]$$

$$= \frac{594966}{1048576} = 0.5674$$

**(ii)** Since $np = 20(\frac{1}{2}) = 10 > 5$
and $nq = 20(1/2) = 10 > 5$
So we will use the normal distribution to approximate the binomial distribution.
Now $\mu = np = 20(\frac{1}{2}) = 10$

$\sigma = \sqrt{npq} = \sqrt{20(0.5)(0.5)} = 2.24$

For the normal approximation, the interval of discrete value $10 \leq x \leq 14$ is replaced by the interval $9.5 \leq x \leq 14.5$
we compute the z-values as below

At $x = 9.5$, we find $z = \frac{9.5-10}{2.24} = -0.22$
and
at $x = 14.5$, we get $z = \frac{14.5-10}{2.24} = 2.01$

**Solved by**
**Labeeb Ahmad**

$P(10 \leq x \leq 14) = P(-0.22 \leq z \leq 2.01)$

$= P(-0.22 \leq z \leq 0) + P(0 \leq z \leq 2.01)$

Now by using table (area under the unit normal curve)

$= 0.0871 + 0.4778$

$= 0.5649$

Hence the Probability is $0.5649$.

## Q7(a) Given that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \, Exp^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \quad -\infty < x < \infty$$

Find the distribution of $y = x^2$

**Solution:-** The distribution of $x$ is

$$df(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} dx, \quad -\infty < x < \infty$$

let $y = x^2$

So that $x = \sqrt{y}$

Solved by Labeeb Ahmad

$$dx = \frac{1}{2\sqrt{y}} \cdot dy$$

Then the distribution of $y$ is

$$df = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\sqrt{y}-\mu}{\sigma}\right]^2} \frac{1}{2\sqrt{y}} \cdot dy \quad \text{--- (1)}$$

let $z = \frac{1}{2}\left[\frac{\sqrt{y}-\mu}{\sigma}\right]^2$

$$2z = \left[\frac{\sqrt{y}-\mu}{\sigma}\right]^2$$

$$\sigma\sqrt{2z}+\mu=\sqrt{y}$$

$$\left[\sigma\sqrt{2z}+\mu\right]^2=y$$

Differentiating

$$2\left[\sigma\sqrt{2z}+\mu\right]\left[\sigma\frac{1}{2\sqrt{2z}}.2.dz\right]=dy$$

Substitue these values in ①

$$df=\frac{1}{\sigma\sqrt{2\pi}}e^{-z}\frac{2\left[\sigma\sqrt{2z}+\mu\right]\left[\sigma(2z)^{-\frac{1}{2}}.dz\right]}{2\left[\sigma\sqrt{2z}+\mu\right]}$$

$$=\frac{1}{\sqrt{2\pi}}\,e^{-z}\,\frac{z^{1/2-1}}{\sqrt{2}}.dz$$

$$=\frac{1}{2}\left[\frac{1}{\Gamma(1/2)}\cdot e^{-z}\,z^{\frac{1}{2}-1}.dz\right]$$

$$0\leq z<\infty$$

This is Gamma variable with parameter $\frac{1}{2}$.

Solved by
Labeeb Ahmad

**Q7 (b)** If $x_1$ and $x_2$ are the standardized independent Normal Variates find the distribution

$$W = \frac{x_1}{x_2}$$

**Solution :-**

As $x_1 \sim N(0,1)$

$$f(x_1) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \qquad -\infty < x_1 < \infty$$

And $x_2 \sim N(0,1)$

$$f(x_2) = \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2} \qquad -\infty < x_2 < \infty$$

The joint distribution of $x_1$ and $x_2$ is

$$f(x_1, x_2) = f(x_1) \cdot f(x_2) \qquad (\because x_1, x_2 \text{ are independent})$$

$$f(x_1, x_2) = \left[\frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}\right]\left[\frac{1}{\sqrt{2\pi}} e^{-x_2^2/2}\right]$$

$$f(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2 + x_2^2)}$$

Solved by
Labeeb Ahmad

Given $W = \frac{x_1}{x_2}$ and let $W_1 = x_2$

$$W = \frac{x_1}{W_1} \quad \text{or} \quad W W_1 = x_1$$

Now we transform $x_1, x_2$ into two new variable $W$ and $W_1$.
There for we use Jacobian Transformation

And $$J\frac{(x_1, x_2)}{(W, W_1)} = \begin{vmatrix} \frac{\partial x_1}{\partial W} & \frac{\partial x_1}{\partial W_1} \\ \frac{\partial x_2}{\partial W} & \frac{\partial x_2}{\partial W_1} \end{vmatrix}$$

$$J = \begin{vmatrix} W_1 & W \\ 0 & 1 \end{vmatrix} = W_1 - (0)(W)$$

$$J = W_1$$

$x_1 = W W_1$, $x_2 = W_1$

So $f(W, W_1) = f(x_1, x_2)\ |J|$

$$f(W, W_1) = \frac{1}{2\pi} e^{-\frac{1}{2}[(W W_1)^2 + W_1^2]} \cdot (W_1)$$

Solved by
Labeeb Ahmad

Pg = 120

$$f(W, W_1) = \frac{1}{2\pi} e^{-\frac{W_1^2}{2}\left[1+W^2\right]} \cdot (W_1)$$

Because we find the distribution of "W"

there for $f(W) = \int_{-\infty}^{\infty} f(W, W_1) \cdot dW_1$

$$f(W) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{W_1^2}{2}(1+W^2)} \cdot W_1 \cdot dW_1$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{W_1^2}{2}(1+W^2)} \cdot \frac{-(1+W^2)}{-(1+W^2)} W_1 \cdot dW_1$$

$$= \frac{1}{2\pi} \cdot \frac{1}{-(1+W^2)} \int_{-\infty}^{\infty} e^{-\frac{W_1^2}{2}(1+W^2)} -(1+W^2) W_1 dW_1$$

$$= \frac{1}{-2\pi(1+W^2)} \left[ e^{-\frac{W_1^2}{2}(1+W^2)} \right]_{-\infty}^{\infty}$$

$$= \frac{-1}{2\pi(1+W^2)} \left[ e^{-\infty} - e^{-\infty} \right]$$

$$f(W) = \frac{-1}{2\pi(1+W^2)} \left[ 0 \right] = 0$$

Solved by
Labeeb Ahmad

A/2014



# Q8(a) Student "t" distribution

Derive the distribution of student "t" and obtain $\mu_{2r}$

**Solution:** Let z be a standard normal variable and u be a $\chi^2$-dist. Then z and u are independent variable.

Then there joint probability function is

$$dF = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{e^{-\frac{u}{2}} u^{\frac{n}{2}-1}}{2^{n/2}\Gamma(n/2)} dz\, du$$

Let $t = \frac{z}{\sqrt{u/n}}$ and $u = w$

Then $J = \frac{\partial(z, u)}{\partial(t, w)} = \begin{vmatrix} \frac{\partial z}{\partial t} & \frac{\partial z}{\partial w} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial w} \end{vmatrix}$

Solved by
Labeeb Ahmad

$= \begin{vmatrix} \sqrt{w/n} & \frac{1}{2\sqrt{w/n}} \\ 0 & 1 \end{vmatrix}$

$|J| = \sqrt{\frac{w}{n}}$

**Solved by**
**Labeeb Ahmad**

Now we transform z and v into two new variable t and w. There for we apply Jacobian transformation

Then the joint p-d-f of t and w is given by

$$df(t,w) = \frac{e^{-\frac{1}{2}\left[t^2\frac{w}{n}+w\right]}}{\sqrt{2\pi}\cdot 2^{\frac{n}{2}}\Gamma(\frac{n}{2})}(w)^{\frac{n}{2}-1}\sqrt{\frac{w}{n}}\,dw\,dt$$

$$= \frac{e^{-\frac{w}{2}\left[\frac{t^2}{n}+1\right]}\; w^{\frac{n}{2}-1+\frac{1}{2}}}{\sqrt{2\pi}\cdot 2^{n/2}\Gamma(n/2)}\frac{1}{\sqrt{n}}\,.\,dw\,.\,dt$$

$$df(t,w) = \frac{e^{-\frac{w}{2}\left[1+\frac{t^2}{n}\right]}}{\sqrt{2n\pi}\cdot 2^{n/2}\Gamma(n/2)}\cdot w^{\frac{n}{2}-\frac{1}{2}}\,.\,dw\,.\,dt$$

Now integrating both side w.r.t "w"

$$df(t) = \frac{dt}{\sqrt{2n\pi}\; 2^{n/2}\Gamma(n/2)}\underbrace{\int_0^\infty e^{-\frac{w}{2}\left(1+\frac{t^2}{n}\right)} w^{\frac{n-1}{2}}\,.\,dw}_{I_1}$$

Take $I_1$ and

put $\frac{w}{2}\left(1+\frac{t^2}{n}\right) = \theta$

$$\frac{w}{2}\left[\frac{n+t^2}{n}\right] = \theta$$

$$w = \frac{2n\theta}{n+t^2}$$

Solved by
Labeeb Ahmad

$$dw = \frac{2n}{n+t^2}\cdot d\theta$$

$$\left\{\begin{array}{l}\theta \to 0 \text{ when } w \to 0 \\ \theta \to \infty \text{ when } w \to \infty\end{array}\right.$$

putting these in $I_1$

$$I_1 = \int_0^\infty e^{-\theta}\left(\frac{2n\theta}{n+t^2}\right)^{\frac{n-1}{2}}\cdot\left(\frac{2n}{n+t^2}\right)d\theta$$

$$I_1 = \int_0^\infty e^{-\theta}\left(\frac{2n}{n+t^2}\right)^{\frac{n+1}{2}}\theta^{\frac{n}{2}-\frac{1}{2}}\cdot d\theta$$

$$I_1 = \left(\frac{2n}{n+t^2}\right)^{\frac{n+1}{2}}\int_0^\infty e^{-\theta}\,\theta^{\frac{n}{2}+\frac{1}{2}-1}\cdot d\theta$$

$$I_1 = \left[\frac{2n}{n+t^2}\right]^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)$$

Putting these in $f(t)$

$$df(t) = \frac{dt}{\sqrt{2n\pi}\; 2^{\frac{n}{2}} \Gamma(n/2)} \cdot \left[\frac{2n}{n+t^2}\right]^{\frac{n+1}{2}} \Gamma\left(\frac{n}{2}+\frac{1}{2}\right)$$

$$= \frac{\left(\frac{n+t^2}{2n}\right)^{-\frac{1}{2}(n+1)} \Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{2^{n/2}\sqrt{2}\cdot\sqrt{n}\sqrt{\pi}\;\Gamma(n/2)}\, dt$$

$$df(t) = \frac{\left(\frac{1}{2}\right)^{-\frac{1}{2}(n+1)} \left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)} \Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{2^{\frac{n}{2}+\frac{1}{2}}\sqrt{n}\;\Gamma(1/2)\;\Gamma(n/2)}\, dt$$

And

$$f(t) = \frac{1}{\beta\left(\frac{n}{2},\frac{1}{2}\right)\sqrt{n}} \cdot \left(1+t^2/n\right)^{-\frac{1}{2}(1+n)}$$

which is a p-d-f of student $t$- Distribution.

**Solved by**
**Labeeb Ahmad**

**For $\mu_{2r}$** = See paper A/2015 Q7(b) at Page "**75**"

A/2014 ; A/2006 , A/2013

**Q8(b)** Show that the chi-square distribution tends to normality as $n \to \infty$.

**Solution:-** The m-g-f of $\chi^2$ is

$$M_0(t) = (1-2t)^{-n/2}$$

Let us consider chi-square standard variable.

$$z = \frac{x^2 - \mu}{\sigma} = \frac{\chi^2 - n}{\sqrt{2n}} \quad \text{where } \mu = n \ \& \ \sigma = \sqrt{2n}$$

Then its m-g-f about mean is

$$M(t) = E(e^{tz})$$

$$= E\left(e^{t\left[\frac{x-\mu}{\sigma}\right]}\right)$$

$$= E\left(e^{tx/\sigma} \cdot e^{-t\mu/\sigma}\right)$$

$$= e^{-t\mu/\sigma} E\left(e^{\frac{t}{\sigma}x}\right)$$

Solved by
Labeeb Ahmad

$$= e^{-t\mu/\sigma} \cdot M_0(t/\sigma)$$

$$M(t) = e^{\frac{-t\mu}{\sigma}} \left(1 - 2\frac{t}{\sigma}\right)^{-n/2}$$

As $\mu = n$ & $\sigma = \sqrt{2n}$

There for

$$M(t) = e^{-tn/\sqrt{2n}} \left[1 - \frac{2t}{\sqrt{2n}}\right]^{-n/2}$$

**Solved by**
**Labeeb Ahmad**

Taking ln both side

$$\ln M(t) = \ln\left[e^{-tn/\sqrt{2n}} \cdot \left(1 - \frac{2t}{\sqrt{2n}}\right)^{-\frac{n}{2}}\right]$$

$$= \frac{-tn}{\sqrt{2n}} \ln e - \frac{n}{2} \ln\left[1 - \frac{2t}{\sqrt{2n}}\right]$$

$$= \frac{-tn}{\sqrt{2n}} - \frac{n}{2}\left[1 + \left(-\frac{2t}{\sqrt{2n}}\right)\right]$$

using expension

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

$$= \frac{-tn}{\sqrt{2n}} - \frac{n}{2}\left[-\frac{2t}{\sqrt{2n}} - \frac{4t^2}{4n} - \frac{8t^3}{6n\sqrt{2n}} - \cdots\right]$$

$$= -\frac{tn}{\sqrt{2n}} + \frac{tn}{\sqrt{2n}} + \frac{t^2}{2} + \text{higher power of } \frac{1}{n}$$

$$\ln M(t) = t^2/2 + \text{higher power of } 1/n$$

Taking $\lim_{n \to \infty}$ both side

$$\lim_{n \to \infty} \ln M(t) = \lim_{n \to \infty}\left[\frac{t^2}{2} + \text{higher power of } \frac{1}{n}\right]$$

$$= t^2/2$$

$\lim_{n \to \infty} M(t) = e^{t^2/2}$ which is m.g.f of the standard normal r.v.

## Q9(a) – see A/2015 Q8(a) at Page "79"

## Q9(b) A/2014

Derive the formula for calulating Rank Correlation between x and y and also calulate the pearson product Correlation Coefficient from the following.

X ⇒ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Y ⇒ 2, 4, 3, 1, 7, 5, 8, 10, 6, 9

**Solution:-** Derivation of Rank Correlation

Let a set of n objects be ranked with respect to character A as $x_1, x_2 \cdots x_n$ and according to character B as $y_1, y_2 — y_n$ we assume that no two or more objects are given the same ranks then obviously $x_i$ and $y_i$ are some two numbers from 1 to n.

Since both $x_i$ and $y_i$ are the first n natural numbers, therefor we have

Solved by
Labeeb Ahmad

$$\sum_{i=1}^{n} x = \sum_{i=1}^{n} y = \sum_{i=1}^{n} (i) = 1+2+3 — n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} x^2 = \sum_{i=1}^{n} y^2 = \sum_{i=1}^{n} (i)^2 = 1^2 + 2^2 + 3^2 + \cdots n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - \frac{(\Sigma y_i)^2}{n}$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{1}{n}\left[\frac{n^2(n+1)^2}{4}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3} - \frac{(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n-1}{6}\right]$$

$$= \frac{n(n^2-1)}{12}$$

**Solved by Labeeb Ahmad**

Let $d_i$ denote the difference in ranks assigned to the ith object i-e $d_i = x_i - y_i$

Then $\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (x_i - y_i)^2$

$$\sum_{i=1}^{n} d_i = \sum (x_i^2 + y_i^2 - 2x_i y_i)$$

$$= \sum x_i^2 + \sum y_i^2 - 2\sum x_i y_i$$

substituting for $\sum x_i^2$ and $\sum y_i^2$, we get

$$\sum_{i=1}^{n} d_i^2 = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)(2n+1)}{6} - 2\sum x_i y_i$$

$$\sum d_i^2 = \frac{n(n+1)(2n+1)}{3} - 2\sum x_i y_i$$

$$2\sum x_i y_i = \frac{n(n+1)(2n+1)}{3} - \sum d_i^2$$

$$\sum x_i y_i = \frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\sum d_i^2$$

The product moment Co-efficient of correlation between two sets of raking is

$$r = \frac{\sum xy - \frac{(\sum x)(\sum y)}{n}}{\sqrt{\left[\sum x^2 - \frac{(\sum x)^2}{n}\right]\left[\sum y^2 - \frac{(\sum y)^2}{n}\right]}}$$

**Solved by Labeeb Ahmad**

substitution gives

$$\frac{\left[\frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\sum d_i^2\right] - \left[\frac{n(n+1)}{4}\right]^2}{\sqrt{\left[\frac{n(n+1)(2n+1)}{6} - \frac{1}{n}\left(\frac{n^2(n+1)^2}{4}\right)\right]^2}}$$

$$\gamma = \frac{\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4}\right] - \frac{1}{2}\sum d_i^2}{\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4}\right]}$$

$$\gamma = \frac{\left[\frac{n(n+1)}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right)\right] - \frac{1}{2}\sum d_i^2}{\left[\frac{n(n+1)}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right)\right]}$$

$$\gamma = \frac{\left[\frac{n(n+1)}{2}\left(\frac{4n+2-3n-3}{6}\right)\right] - \frac{1}{2}\sum d_i^2}{\left[\frac{n(n+1)}{2}\left(\frac{4n+2-3n-3}{6}\right)\right]}$$

$$\gamma = \frac{\left[\frac{n(n+1)(n-1)}{12}\right] - \frac{1}{2}\sum d_i^2}{\left[\frac{n(n+1)(n-1)}{12}\right]}$$

$$\gamma = \frac{\left[n(n^2-1)/12\right] - \frac{1}{2}\sum d_i^2}{\left[n(n^2-1)/12\right]}$$

$$\gamma = 1 - \frac{\frac{1}{2}\sum d_i^2}{\frac{n(n^2-1)}{12\ 6}}$$

$$\gamma = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

Solved by
Labeeb Ahmad

Now



| X | Y | $x-\bar{x}$ | $(x-\bar{x})^2$ | $y-\bar{y}$ | $(y-\bar{y})^2$ | $(x-\bar{x})(y-\bar{y})$ |
|---|---|---|---|---|---|---|
| 1 | 2 | -4.5 | 20.25 | -3.5 | 12.25 | 15.75 |
| 2 | 4 | -3.5 | 12.25 | -1.5 | 2.25 | 5.25 |
| 3 | 3 | -2.5 | 6.25 | -2.5 | 6.25 | 6.25 |
| 4 | 1 | -1.5 | 2.25 | -4.5 | 20.25 | 6.75 |
| 5 | 7 | -0.5 | 0.25 | 1.5 | 2.25 | -0.75 |
| 6 | 5 | 0.5 | 0.25 | -0.5 | 0.25 | -0.25 |
| 7 | 8 | 1.5 | 2.25 | 2.5 | 6.25 | 3.75 |
| 8 | 10 | 2.5 | 6.25 | 4.5 | 20.25 | 11.25 |
| 9 | 6 | 3.5 | 12.25 | 0.5 | 0.25 | 1.75 |
| 10 | 9 | 4.5 | 20.25 | 3.5 | 12.25 | 15.75 |
| 55 | 55 | 0 | 82.5 | 0 | 82.5 | 65.5 |

$$\bar{X} = \frac{\sum X}{n} = \frac{55}{10} = 5.5$$

$$\bar{Y} = \frac{\sum Y}{n} = \frac{55}{10} = 5.5$$

**Solved by Labeeb Ahmad**

$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2 \sum (y-\bar{y})^2}}$$

$$r = \frac{65.5}{\sqrt{(82.5)(82.5)}} = \frac{65.5}{82.5} = 0.794$$

# Alternatively:

| X | Y | $X^2$ | $Y^2$ | XY |
|---|---|---|---|---|
| 1 | 2 | 1 | 4 | 2 |
| 2 | 4 | 4 | 16 | 8 |
| 3 | 3 | 9 | 9 | 9 |
| 4 | 1 | 16 | 1 | 4 |
| 5 | 7 | 25 | 49 | 35 |
| 6 | 5 | 36 | 25 | 30 |
| 7 | 8 | 49 | 64 | 56 |
| 8 | 10 | 64 | 100 | 80 |
| 9 | 6 | 81 | 36 | 54 |
| 10 | 9 | 100 | 81 | 90 |
| 55 | 55 | 385 | 385 | 368 |
| | | | | |

$$r = \frac{\Sigma XY - (\Sigma X)(\Sigma Y)/n}{\sqrt{\left[\Sigma X^2 - (\Sigma X)^2/n\right]\left[\Sigma Y^2 - (\Sigma Y)^2/n\right]}}$$

$$r = \frac{368 - (55)(55)/10}{\sqrt{\left[385 - (55)^2/10\right]\left[385 - (55)^2/10\right]}}$$

$$r = \frac{65.5}{\sqrt{(82.5)(82.5)}} = \frac{65.5}{82.5}$$

$$r = 0.794$$

**Solved by**
**Labeeb Ahmad**

# PAPER A/2013

**Solved by**
**Labeeb Ahmad**

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**Q1 (a)** A Coin is biased so that the probability showing head is $\frac{3}{4}$

(i) Find the probability obtaining at least one head when coin is tossed n times

(ii) How many times the coin is tossed so that the probability of obtaining at least one head is greater than 0.98?

**Solution:-** Here $P(\text{a head appears}) = \frac{1}{4}$

And $P(\text{no head appears}) = \frac{3}{4}$

(i) Let A be the event at least one head is obtained when the coin is tossed n times and $\bar{A}$ is the event that no head is obtained.

Then by the law of complementation we have

$$P(A) = 1 - P(\overline{A})$$

Solved by
Labeeb Ahmad

$$= 1 - \left(\frac{3}{4}\right)^n$$

(ii) Let the coin be tossed $n$ times to obtain the probability of at least one head greater than 0.98

Then $1 - (3/4)^n \geq 0.98$

i.e $\left(\frac{3}{4}\right)^n \leq 0.02$

Taking log, we have

$$n \log\left(\frac{3}{4}\right) \leq \log(0.02)$$

Dividing both side by log(3/4) and reversing the inequality, sign $\log\left(\frac{3}{4}\right)$ is negative, we have

$$n \geq \frac{\log(0.02)}{\log(3/4)} \geq \frac{\bar{2}.3010}{\bar{1}.8751} \geq \frac{-1.6990}{-0.1249} \geq 13.6$$

So least $n = 14$.

Hence the coin should be tossed 14 times so the probability of obtaining atleast one head is greater than 0.98.

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# Q1(b) Stat and Prove the Addition Law of three events A, B and C

**Statment** If A, B and C are any three events defined in a sample space S, then

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B) - P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

**Proof** $P(A\cup B\cup C)$ 

Solved by Labeeb Ahmad

$= P[A\cup(B\cup C)]$

using $P(A\cup B) = P(A)+P(B)-P(A\cap B)$

$= P(A)+P(B\cup C)-P(A\cap(B\cup C))$

$= P(A)+P(B)+P(C)-P(B\cap C) - P(A\cap(B\cup C))$ —— (1)

Take $P[A\cap(B\cup C)] = P[(A\cap B)\cup(A\cap C)]$

$= P(A\cap B)+P(A\cap C)-P(A\cap B\cap C)$

putting this in (1)

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(B\cap C) - P(A\cap B)-P(A\cap C)+P(A\cap B\cap C)$$

Hence Prove

**Q2(a)** Derive c-g-f of the binomial Distribution and write down its first four moments.

**Solution :-** For c-g-f first we find m-g-f

$$M_o(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x q^{n-x}$$

$$M_o(t) = (q + pe^t)^n$$

For C-g-f Taking ln both side

$$\ln M_o(t) = \ln (q + pe^t)^n$$

$$= n \ln (q + pe^t)$$

As we know that By Taylor series

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \text{——}$$

$$\ln M_o(t) = n \ln \left[ q + p\left\{1 + t + \frac{t^2}{2!} + \frac{t^3}{3} + \text{——}\right\}\right]$$

Solved by
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$$K(t) = \ln M_o(t) = n \ln\left[q + p + pt + \frac{pt^2}{2!} + \frac{pt^3}{3!} + \right]$$

$$K(t) = n \ln\left[1 - p + p + pt + \frac{pt^2}{2!} + \frac{pt^3}{3!} + \text{---}\right]$$

$$K(t) = n \ln\left[1 + \left\{pt + \frac{pt^2}{2!} + \frac{pt^3}{3!} + \text{---}\right\}\right]$$

As we know By Taylor series

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \text{------}$$

$$K(t) = n\left(\left[pt + \frac{pt^2}{2!} + \frac{pt^3}{3!} + \text{----}\right] - \frac{1}{2}\left[pt + \frac{pt^2}{2!} + \frac{pt^3}{3!} + \text{---}\right]^2 + \frac{1}{3}\left[pt + \frac{pt^2}{2!} + \frac{pt^3}{3!} + \text{---}\right]^3 - \frac{1}{4}\left[pt + \frac{pt^2}{2!} +\right]^4\right)$$

As $K(t) = K_1 t + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + \text{---}$

Compare the co-efficient of like power of "t", we get

Solved by
Labeeb Ahmad

$K_1 = \mu_1 = np$

$K_2 = \mu_2 = np - np^2$

$= np(1-p)$

$= npq \quad \because 1 - p = q$

$K_3 = \mu_3$

$K_3 = np - 3np^2 + 2np^3$

$= np(1 - 3p + 2p^2)$

$= np(1-p)(1-2p) = npq(1-2p)$

$K_4 = np - 3np^2 + 12np^3 - 6np^4 - 4np^2$

$= np - 7np^2 + 12np^3 - 6np^4$

$= np[1 - 7p^2 + 12p^2 - 6p^3]$

$= np(-1+p)(-6p^2 + 6p - 1)$

$= np(1-p)(6p^2 - 6p + 1)$

$= npq[(6p)(p-1) + 1]$

$= npq[-6pq + 1]$

$K_4 = npq[1 - 6pq]$

$K_4 = \mu_4 - 3\mu_2^2$

$\mu_4 = K_4 + 3\mu_2^2$

$\mu_4 = npq[1 - 6pq] + 3(npq)^2$

$\mu_4 = npq[3npq + 1 - 6pq]$

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**Q2(b)** Show that for Poisson distribution

$$\mu_{r+1} = \lambda\left[r\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right]$$

**Solution:-**

See Paper A/2014 Question 4(b) at page "101"

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**Q2(c)** A Box Contains 5 blue balls and 8 green balls; 3 ball are selected at random.
Find the Probabilities of getting 2 green balls
(i) with replacment
(ii) without replacment

**Solved by Labeeb Ahmad**

**Solution:-**

| Blue | , | Green | = | Total |
|---|---|---|---|---|
| 5 | + | 8 | = | 13 |

**(i) With replacement**

Here x is a Binomial r.v

$n=3$, $p=$ Prob of green $= 8/13$

$q=$ Prob of blue $= 5/13$

$$P(X=2) = \binom{3}{2}\left(8/13\right)^2\left(5/13\right)^1$$
$$= 0.4369595$$

**(ii) without replacment**

Here x is a hypergeometric r-v

$n=3$ $\quad N = 8+5 = 13$

$K=8$ $\quad x=2$ $\quad N-K=5$ $\quad n-x=2$

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} = \frac{\binom{8}{2}\binom{5}{1}}{\binom{13}{3}} = \frac{140}{286}$$
$$= 0.4895104$$

**Q3(a)** Obtain the moment generating function of the Possion Distribution and show that all its cumulants are equal.

## Solution :-

By Definition

$M_o(t) = E(e^{tx})$

$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\mu}\mu^x}{x!}$

$= e^{-\mu} \sum_{x=0}^{\infty} \frac{e^{tx}\mu^x}{x!}$

Solved by
Labeeb Ahmad

$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!}$

$= e^{-\mu}\left[1 + \frac{(\mu e^t)^1}{1!} + \frac{(\mu e^t)^2}{2!} + \frac{(\mu e^t)^3}{3!} + \text{---}\right]$

$= e^{-\mu}\left[e^{\mu e^t}\right]$

$M_o(t) = e^{\mu e^t - \mu} = e^{\mu[e^t - 1]}$

For c-g-f Taking ln both side

$\ln[M_o(t)] = \ln[e^{\mu(e^t-1)}]$

$= \mu(e^t - 1)$

$k(t) = \mu\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \text{-----} - 1\right]$

$k_1(t) + k_2\frac{t^2}{2!} + k_3\frac{t^3}{3!} + \text{---} = \mu\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \text{---}\right]$

Compering power of t's

$K_1 = K_2 = K_3 = K_4 \text{ --- } K_n = \mu$

## Q3(b)

The number of minuts that a flight from London to Karachi is early or late is a random variable whose probability density function is given by

$$f(x) = \begin{cases} \frac{1}{200}(36-x^2) & ; -6<x<6 \\ 0 & \text{otherwise} \end{cases}$$

where the negative values are indicator of flight being early and positive values are indicater of being its late.
Find the probabilities that one of these flight will be

(i) at least 2 minuts early
(ii) at least one minut late
(iii) any where from 1 to 3 minuts early

Solved by
Labeeb Ahmad

## Solution :-

Given that

$$f(x) = \begin{cases} \frac{1}{200}(36-x^2), & -6<x<6 \\ 0 & ; \text{elsewhere} \end{cases}$$

**(i)** At least 2 minuts early

$P(-6 \leq x \leq -2)$

$$= \int_{-6}^{-2} \frac{36 - x^2}{200} \cdot dx$$

$$= \frac{1}{200}\left[36x - \frac{x^3}{3}\right]_{-6}^{-2}$$

$$= \frac{1}{200}\left[36(-2) - \frac{(-2)^3}{3} - 36(-6) + \frac{(-6)^3}{3}\right]$$

$$= \frac{1}{200}\left[-72 + \frac{8}{3} + 216 - \frac{216}{3}\right]$$

$$= \frac{1}{200}\left[\frac{224}{3}\right] = \frac{224}{600}$$

$$= 0.3733$$

**(ii)** At least one minute late

$P(x \geq 1)$

$$= \int_{1}^{6} \frac{1}{200}(36 - x^2) \cdot dx$$

$$= \frac{1}{200}\left[36x - \frac{x^3}{3}\right]_{1}^{6}$$

Solved by Labeeb Ahmad

$$= \frac{1}{200}\left[36(6) - \frac{6^3}{3} - 36(1) + \frac{1^3}{3}\right]$$

$$= \frac{1}{200}\left[216 - \frac{216}{3} - 36 + \frac{1}{3}\right]$$

$$= \frac{1}{200}\left[\frac{325}{3}\right]$$

$$= 0.54166$$

**(iii)** any where from 1 to 3 minuts early.

$P(-3 \leq X \leq -1)$

$$= \int_{-3}^{-1} f(x) \cdot dx = \int_{-3}^{-1} \frac{1}{200}(36 - x^2) \cdot dx$$

$$= \frac{1}{200}\left[36x - \frac{x^3}{3}\right|_{-3}^{-1}$$

$$= \frac{1}{200}\left[36(-1) - \frac{(-1)^3}{3} - 36(-3) + \frac{(-3)^3}{3}\right]$$

$$= \frac{1}{200}\left[-36 + \frac{1}{3} + 108 - \frac{27}{3}\right]$$

$$= \frac{1}{200}\left[\frac{190}{3}\right]$$

$$= \frac{190}{600} = 0.31666$$

Solved by
Labeeb Ahmad

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**Q4(a)** Find the joint probability density function of two random variables X and Y. whose joint density is given by

$$F(x,y) = 1 - e^{-x} - e^{-y} + e^{-x-y}$$

$$x > 0 \ , \ y > 0$$

then find $P[(X+Y) > 3]$.

**Solution** :- Given

$$F(x,y) = 1 - e^{-x} - e^{-y} + e^{-x-y}$$

As we know that $\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y)$

So $\frac{\partial^2}{\partial x \partial y}(1 - e^{-x} - e^{-y} + e^{-x-y})$

$$= \frac{\partial}{\partial x}\left[0 - 0 - e^{-y}(-1) + e^{-x-y}(-1)\right]$$

$$= - e^{-x-y}(-1)$$

$$= e^{-x-y}$$

Solved by
Labeeb Ahmad

Hence the density function is

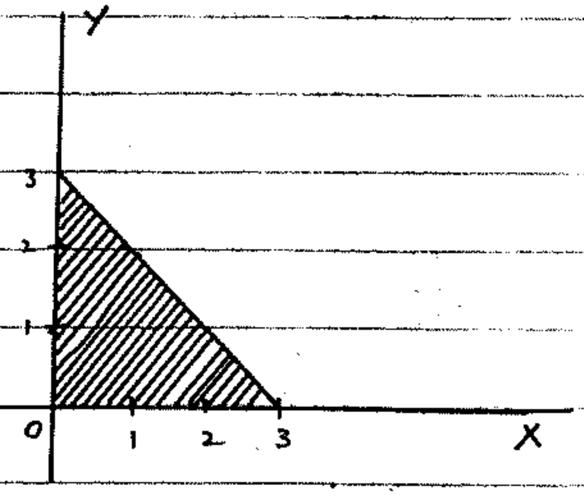
$f(x,y) = e^{-x-y}$

Now we find $P(x+y > 3)$

$P[(X+Y)>3] = 1 - P(X+Y \leq 3)$ —— (i)

First we find

$P(X+Y \leqslant 3)$




So

$$P[(X+Y)\leqslant 3] = \int_0^3\int_0^{3-x} f(x,y)\,dy\,dx$$

$$= \int_0^3\int_0^{3-x} e^{-x-y}\,dy.dx$$

$$= \int_0^3 e^{-x}\left\{\int_0^{3-x} e^{-y}.dy\right\}dx$$

Solved by
Labeeb Ahmad

$$= \int_0^3 e^{-x}\left|-e^{-y}\right|_0^{3-x} dx$$

$$= \int_0^3 e^{-x}\left(-e^{-(3-x)} + e^{0}\right).dx$$

$$= \int_0^3 e^{-x}\left(-e^{-3+x} + 1\right).dx$$

$$= -\int_0^3 \left(e^{-3+x-x} - e^{-x}\right)dx$$

$$= -\int_0^3 e^{-3} - e^{-x}.dx$$

$$= -\left[e^{-3}|x|_0^3 - \left|\frac{e^{-x}}{-1}\right|_0^3\right]$$

$$= -\left[3e^{-3} + e^{-3} - e^{0}\right]$$

$$= -[4e^{-3} - 1]$$

$$= 1 - 4e^{-3}$$

put in (i)

$$P(X+Y>3) = 1-(1-4e^{-3})$$

$$= 1 - 1 + 4e^{-3}$$

$$= 4e^{-3}$$

$$= 0.19914$$

# Q4(b) Stat and Prove

## Chebyshev's Inequality

**Statment**

If $X$ is a r-v having mean $\mu$ & variance $\sigma^2 > 0$, and $K$ is any positive constant the probability that a value of $X$ falls within $K$ standard deviation of the mean is at least $\left[1-\frac{1}{K^2}\right]$

That is

$$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1-\frac{1}{K^2}$$

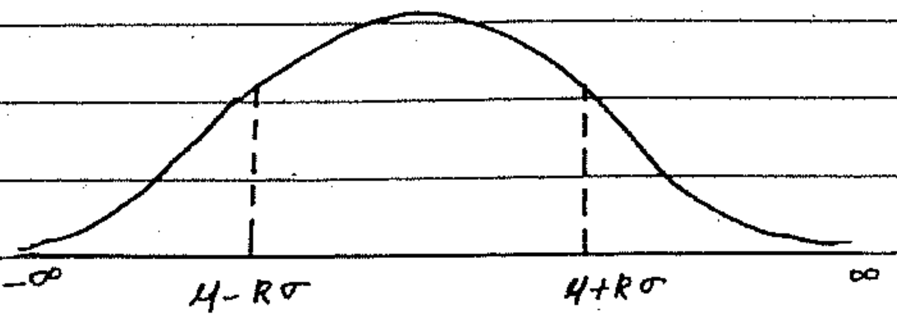
or $P[\,|X-\mu| \geq K\sigma\,] \leq \frac{1}{K^2}$

Solved by Labeeb Ahmad

**Proof:-**

By definition, we have

$$\sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$




Dividing the rang into 3 disjoint parts

$(-\infty, \mu - k\sigma)$, $(\mu - k\sigma, \mu + k\sigma)$ and $(\mu + k\sigma, \infty)$

we have

$$\sigma^2 = \int_{-\infty}^{\mu - k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu - k\sigma}^{\mu + k\sigma} (x-\mu)^2 f(x)\,dx$$

$$+ \int_{\mu + k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

Solved by
Labeeb Ahmad

Dropping the middle term, we get

$$\sigma^2 \geq \int_{-\infty}^{\mu - k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu + k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

Since $|x-\mu| \geq k\sigma$, we have

$(x-\mu)^2 \geq k^2\sigma^2$ wherever $x \geq \mu + k\sigma$

or $x \leq \mu - k\sigma$.

It there for follows that

$$\sigma^2 \geq \int_{-\infty}^{\mu - k\sigma} k^2\sigma^2 f(x)\,dx + \int_{\mu + k\sigma}^{\infty} k^2\sigma^2 f(x)\,dx$$

$$\text{or} \quad \int_{-\infty}^{\mu - k\sigma} f(x)\,dx + \int_{\mu + k\sigma}^{\infty} f(x)\,dx \leq \frac{1}{k^2}$$

Hence $P(\mu - k\sigma < x < \mu + k\sigma) = \int_{\mu - k\sigma}^{\mu + k\sigma} f(x)\,dx \geq 1 - \frac{1}{k^2}$

A/2013



**Q5(a)** If the joint distribution of $X_1, X_2$ is given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & x_1 > 0,\ x_2 > 0 \\ 0 & \text{else where} \end{cases}$$

Find the joint density of

$Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_1}{X_1+X_2}$

**Solution :-**

$$Y_2 = \frac{X_1}{Y_1} \quad \text{or} \quad X_1 = Y_1 Y_2$$

And $Y_1 = X_1 + X_2$

put the value of $X_1$

$$Y_1 = Y_1 Y_2 + X_2$$

$$Y_1 - Y_1 Y_2 = X_2$$

$$X_2 = Y_1(1 - Y_2)$$

Solved by
Labeeb Ahmad

$$J\frac{(X_1, X_2)}{(Y_1, Y_2)} = \begin{vmatrix} \dfrac{\partial X_1}{\partial Y_1} & \dfrac{\partial X_1}{\partial Y_2} \\ \dfrac{\partial X_2}{\partial Y_1} & \dfrac{\partial X_2}{\partial Y_2} \end{vmatrix}$$

$$J = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix}$$

$= -y_2 y_1 - (y_1)(1-y_2)$

$= -y_1 y_2 - y_1 + y_1 y_2$ (the $y_1 y_2$ terms are struck through)

$J = -y_1$

$|J| = y_1$ $\quad x_1 = y_1 y_2$, $\quad x_2 = y_1(1-y_2)$

So $f(y_1, y_2) = f(x_1, x_2)\,|J|$

$f(y_1, y_2) = e^{-y_1} \cdot y_1 \;, \quad y_1 > 0$

This is a joint probability density function of $y_1, y_2$

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**Q5(b)** Obtain the moment generating function of the $\chi^2$-Distribution and obtain its mean and variance. Also discuss its symmetry and Kurtosis.

## Solution :-

By Definition

$$M_0(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot f(x) \cdot dx$$

$$= \int_0^\infty e^{tx} \frac{x^{\frac{n}{2}-1} e^{-x/2}}{2^{n/2} \Gamma(n/2)} dx$$

$$= \frac{1}{2^{n/2} \Gamma(n/2)} \int_0^\infty x^{\frac{n}{2}-1} e^{tx - \frac{x}{2}} . dx$$

$$= \frac{1}{2^{n/2} \Gamma(n/2)} \int_0^\infty e^{-\frac{x}{2}(1-2t)} x^{\frac{n}{2}-1} . dx$$

Solved by
Labeeb Ahmad

Put $\frac{x}{2}(1-2t) = \theta$

$$dx \frac{(1-2t)}{2} = d\theta$$

And $\theta \to \infty$ when $x \to \infty$

$\theta \to 0$ when $x \to 0$

$$M_o(t) = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty e^{-\theta} \left[\frac{2\theta}{(1-2t)}\right]^{\frac{n}{2}-1} \left(\frac{2}{1-2t}\right) d\theta$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty e^{-\theta} \left(\frac{2}{1-2t}\right)^{\frac{n}{2}} \cdot \theta^{\frac{n}{2}-1} \cdot d\theta$$

$$= \frac{2^{n/2}}{2^{n/2}\Gamma(n/2)\,(1-2t)^{n/2}} \int_0^\infty e^{-\theta}\,\theta^{\frac{n}{2}-1} \cdot d\theta$$

$$= \frac{1}{\Gamma(n/2)\,(1-2t)^{n/2}}\,\Gamma(n/2)$$

**Solved by Labeeb Ahmad**

$$M_o(t) = (1-2t)^{-n/2}$$

Now using Binomial Theorem

$$M_o(t) = 1 + \left(\frac{-n}{2}\right)(-2t) + \frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}-1\right)}{2!}(-2t)^2$$

$$+ \frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}-1\right)\left(-\frac{n}{2}-2\right)}{3!}(-2t)^3 + \frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}-1\right)\left(-\frac{n}{2}-2\right)\left(-\frac{n}{2}-3\right)}{4!}(-2t)^4$$

$$- - - - -$$

$$M_o(t) = 1 + nt + \frac{n(n+2)}{2!}t^2 + \frac{n(n+2)(n+4)}{3!}t^3$$

$$+ \frac{n(n+2)(n+4)(n+6)}{4!}t^4 + \ldots$$

Also $M_0(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \frac{t^4}{4}\mu_4'$

Comparing the coefficient of $\frac{t^r}{r!}$ in $M_0(t)$

$\mu_1' = n \longrightarrow$ **Mean**

$\mu_2' = n(n+2)$

$\mu_3' = n(n+2)(n+4)$

$\mu_4' = n(n+2)(n+4)(n+6)$

Now we find $\mu_1, \mu_2, \mu_3, \mu_4$

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1'^2 = n(n+2) - n^2$

$= n^2 + 2n - n^2$

$= 2n \longrightarrow$ **Variance**

$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

$= n(n+2)(n+4) - 3(n)(n)(n+2) + 2n^3$

$= (n^2+2n)(n+4) - 3n^3 - 6n^2 + 2n^3$

$= n^3 + 4n^2 + 2n^2 + 8n - 3n^3 - 6n^2 + 2n^3$

$\mu_3 = 8n$

**Solved by**
**Labeeb Ahmad**

$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3(\mu_1')^4$

$= n(n+2)(n+4)(n+6) - 4(n)[n(n+2)(n+4)]$
$+ 6n^2[n(n+2)] - 3n^4$
$= (n^2+2n)(n^2+6n+4n+24) - 4n^2[n^2+4n+2n+8]$
$+ 6n^3(n+2) - 3n^4$
$= n^4 + 10n^3 + 24n^2 + 2n^3 + 20n^2 + 48n$
$- 4n^4 - 16n^3 - 8n^3 - 32n^2 + 6n^4 + 12n^3 - 3n^4$

$= 12n^2 + 48n$

**Solved by**
**Labeeb Ahmad**

$\mu_4 = 12n(n+4)$

**Skewness**: $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(8n)^2}{(2n)^3} = \frac{64n^2}{8n^3} = \frac{8}{n}$

**Kurtosis** :- $\beta_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{12n(n+4)}{(2n)^2}$

$= \frac{12n^2+48n}{4n^2} = 3 + \frac{12}{n}$

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**Q6(a)** If $x_r$ and $x_s$ are the rth and sth random variable of sample size n drawn from the finite population $\{C_1, C_2, C_3 \text{ --- } C_N\}$ then prove. $Cov(x_r, x_s) = -\frac{\sigma^2}{N-1}$

**Solution**. See paper A|2015 Q5(b) at page "64"

**Q6(b)** If $\bar{x}$ and $s^2$ are mean and variance of a random sample then show that $t = \frac{\bar{x}-\mu}{s/\sqrt{n}}$ is t-Distribution with (n-1) degree of freedom.

**Proof :-**

t also written as

$$t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}} \cdot \frac{s}{\sigma}} = \frac{(\bar{x}-\mu)/\sigma\sqrt{n}}{\sqrt{\frac{1}{n-1}\sum(x_i-\bar{x})^2} \cdot \frac{1}{\sigma}}$$

$$= \frac{(\bar{x}-\mu)/\sigma\sqrt{n}}{\sqrt{\frac{1}{n-1}\sum\left(\frac{x_i-\bar{x}}{\sigma}\right)^2}} = \frac{Z}{\sqrt{\frac{U}{n-1}}}$$

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where $Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is standard normal r.v

and $U = \sum\left[\frac{x_i-\bar{x}}{\sigma}\right]^2 = \frac{(n-1)s^2}{\sigma^2}$ is $\chi^2$ r.v with (n-1) degree of freedom.

Thus the t-statistic is the quotient of a standard normal variable and the square root of chi-square random variable divided by its degree of freedom.

To find the distribution of "t" we proceeds as follows.

Let z be a standard normal variable and u be a $\chi^2$ - Distribution. Then z & u are independent variable.

then there joint probability function is

$$dF = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{e^{-\frac{u}{2}} u^{\frac{n}{2}-1}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} dz \cdot du$$

Let

$$t = \frac{z}{\sqrt{w/n}} \quad \text{and} \quad u = w$$

$$J = \frac{\partial(z,u)}{\partial(t,w)} = \begin{vmatrix} \frac{\partial z}{\partial t} & \frac{\partial z}{\partial w} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial w} \end{vmatrix}$$

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$$= \begin{vmatrix} \sqrt{w/n} & \frac{1}{2\sqrt{w/n}} \\ 0 & 1 \end{vmatrix}$$

So $|J| = \sqrt{\frac{w}{n}}$

Now we transform z and u into two new variable t and w. There for we apply Jacobian Transformation.

Then the joint p.d.f of "t" an "w" is given by

$$df(t,w) = \frac{e^{-\frac{1}{2}\left[t^2\frac{w}{n}+w\right]}}{\sqrt{2\pi}\;2^{n/2}\,\Gamma n/2}\, w^{\frac{n}{2}-1}\sqrt{\frac{w}{n}}\,dw.dt$$

$$= \frac{e^{-\frac{w}{2}\left[\frac{t^2}{n}+1\right]}}{\sqrt{2\pi}\;2^{n/2}\,\Gamma(n/2)}\, w^{\frac{n}{2}-1+\frac{1}{2}}\,\frac{1}{\sqrt{n}}\,dwdt$$

$$df(t,w) = \frac{e^{-\frac{w}{2}\left[1+\frac{t^2}{n}\right]}}{\sqrt{2n\pi}\;2^{n/2}\,\Gamma(n/2)}\,.\,w^{\frac{n}{2}-\frac{1}{2}}\,.\,dw.dt$$

Now we integrateing both side w.r.t "w"

$$df(t) = \frac{dt}{\sqrt{2n\pi}\cdot 2^{n/2}\,\Gamma(n/2)}\int_0^\infty e^{-\frac{w}{2}\left(1+\frac{t^2}{n}\right)} w^{\frac{n-1}{2}}.dw$$

↓ $I_1$

Take $I_1$ and put

$$\frac{w}{2}\left(1+\frac{t^2}{n}\right)=\theta$$

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$$\frac{w}{2}\left[\frac{n+t^2}{n}\right]=\theta$$

$$w = \frac{2n\theta}{n+t^2}$$

So

$$dw = \frac{2n}{n+t^2}\,d\theta$$

$$\begin{cases} \theta \to 0 & \text{when } \omega \to 0 \\ \theta \to \infty & \text{when } \omega \to \infty \end{cases}$$

Putting these in $I_1$

$$I_1 = \int_0^\infty e^{-\theta} \left[\frac{2n\theta}{n+t^2}\right]^{\frac{n-1}{2}} \left[\frac{2n}{n+t^2}\right] d\theta$$

$$I_1 = \int_0^\infty e^{-\theta} \left[\frac{2n\theta}{n+t^2}\right]^{\frac{n+1}{2}} \theta^{\frac{n}{2}-\frac{1}{2}} \cdot d\theta$$

$$I_1 = \left[\frac{2n}{n+t^2}\right]^{\frac{2n+1}{2}} \int_0^\infty e^{-\theta} \theta^{\frac{n}{2}+\frac{1}{2}-1} \cdot d\theta$$

$$I_1 = \left[\frac{2n}{n+t^2}\right]^{\frac{n+1}{2}} \Gamma\left(\frac{n}{2}+\frac{1}{2}\right)$$

Putting these in $f(t)$

$$d\,f(t) = \frac{dt}{\sqrt{2n\pi}\; 2^{n/2}\; \Gamma\left(\frac{n}{2}\right)} \left[\frac{2n}{n+t^2}\right]^{\frac{n+1}{2}} \Gamma\left[\frac{n}{2}+\frac{1}{2}\right]$$

$$= \frac{\left[\frac{n+t^2}{2n}\right]^{-\frac{1}{2}(n+1)} \Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{2^{n/2}\sqrt{2}\cdot\sqrt{n}\cdot\sqrt{\pi}\;\Gamma(n/2)} dt$$

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$$df(t) = \frac{\left(\frac{1}{2}\right)^{-\frac{1}{2}(n+1)} \left[1+\frac{t^2}{n}\right]^{-\frac{1}{2}(n+1)} \Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{2^{\frac{n}{2}+\frac{1}{2}} \sqrt{n}\, \Gamma(1/2)\, \Gamma(n/2)} dt$$

$$df(t) = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)} \frac{1}{\sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(1+n)} . dt$$

$$df(t) = \frac{1}{\beta\left(\frac{n}{2},\frac{1}{2}\right)} \frac{1}{\sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(1+n)} dt$$

And

$$f(t) = \frac{1}{\beta\left(\frac{n}{2},\frac{1}{2}\right)} \frac{1}{\sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(1+n)}$$

which is a p.d.f of student "t" distribution.

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**Labeeb Ahmad**

**Q7(a)** Find the mode of Gamma and Beta Distribution of 1st kind.

**Solution :-**

Mode :- any value of $x$ for which $f'(x)=0$ and $f''(x)<0$ is called Mode.

Mode for Gamma Distribution

$$f(x) = \frac{1}{\Gamma(n)} x^{n-1} e^{-x}$$

$$f'(x) = \frac{1}{\Gamma(n)} (x^{n-1})(-e^{-x}) + (e^{-x})[(n-1)(x^{n-2})]$$

$$f'(x) = \frac{1}{\Gamma(n)} e^{-x} x^{n-1} \left[-1 + \frac{(n-1)}{x}\right]$$

Equating $f'(x) = 0$

we see $\frac{1}{\Gamma(n)} e^{-x} x^{n-1} \neq 0$

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there for $-1 + \frac{n-1}{x} = 0$

$$\frac{n-1}{x} = 1$$

$$x = n-1 \quad \because (n>1)$$

Now again differentiating

$$f''(x) = \frac{1}{\Gamma(n)}\left[e^{-x} x^{n-1}\right]\left[-\frac{(n-1)}{x^2}\right] + \left[-1 + \frac{n-1}{x}\right]\left[e^{-x} x^{n-1}\left(\frac{n-1}{x} - 1\right)\right]$$

$$= \frac{1}{\Gamma(n)}\left(e^{-x} x^{n-1}\right)\left[-\frac{(n-1)}{x^2} + \left(\frac{n-1}{x} - 1\right)^2\right]$$

put $x = n-1$

$$= \frac{1}{\Gamma(n)}\left(e^{-x} x^{n-1}\right)\left[-\frac{(n-1)}{(n-1)^2} + \left(\frac{n-1}{n-1} - 1\right)^2\right]$$

$$= \frac{1}{\Gamma(n)}\left(e^{-x} x^{n-1}\right)\left[-\frac{1}{(n-1)} + 0\right]$$

$(\because n > 1)$

$$f''(x) = -\frac{1}{\Gamma(n)}\left(e^{-x} x^{n-1}\right)\left(\frac{1}{n-1}\right)$$

$$f''(x) < 0$$

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Thus $x = n-1$ is the mode of Gamma distribution.

*————————*

Now

Mode for Beta Distribution of First kind.

$$f(x) = \frac{1}{B(m,n)} x^{m-1} (1-x)^{n-1} \quad \text{where } m,n>0$$

Differentiate w.r.t "x"

$$f'(x) = \frac{1}{\beta(m,n)} \left(x^{m-1}\right)\left[(n-1)(1-x)^{n-2}(-1)\right] + (1-x)^{n-1}\left[(m-1)x^{m-2}\right]$$

$$= \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} \left[ -\frac{(n-1)}{1-x} + \frac{(m-1)}{x} \right] \quad \text{--- (i)}$$

Equating $f'(x)=0$

we see $\frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} \neq 0$

So $-\frac{(n-1)}{1-x} + \frac{m-1}{x} = 0$

$$\frac{m-1}{x} = \frac{n-1}{1-x}$$

$$\frac{x}{m-1} = \frac{1-x}{n-1}$$

$$\frac{x}{m-1} = \frac{1}{n-1} - \frac{x}{n-1}$$

$$\frac{x}{m-1} + \frac{x}{n-1} = \frac{1}{n-1}$$

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$$x\left[\frac{1}{m-1}+\frac{1}{n-1}\right]=\frac{1}{n-1}$$

$$x\left[\frac{n-1+m-1}{(m-1)(n-1)}\right]=\frac{1}{n-1}$$

$$x=\frac{m-1}{m+n-2} \qquad m>1 \text{ and } n>1$$

Now again Differentiate w.r.t "x" of Equation (i)

$$f''(x)=\frac{1}{\beta(m,n)}\left[x^{m-1}(1-x)^{n-1}\right]\left[-\frac{(n-1)}{(1-x)^2}-\frac{(m-1)}{x^2}\right]+$$

$$\left[-\frac{(n-1)}{1-x}+\frac{m-1}{x}\right]\left[x^{m-1}(1-x)^{n-1}\left(-\frac{(n-1)}{1-x}+\frac{m-1}{x}\right)\right]$$

$$f''(x)=\frac{1}{\beta(m,n)}\left[x^{m-1}(1-x)^{n-1}\right]\left(\frac{(1-n)}{(1-x)^2}+\frac{(1-m)}{x^2}+\left(\frac{1-n}{1-x}+\frac{m-1}{x}\right)^2\right)$$

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Now put $x=\frac{m-1}{m+n-2}$

$$f''(x)=\frac{1}{\beta(m,n)}\left[x^{m-1}(1-x)^{n-1}\right]\left[\frac{n-1}{\left[1-\frac{m-1}{(m+n-2)}\right]^2}+\frac{1-m}{\left(\frac{m-1}{m+n-2}\right)^2}\right]$$

$$+\left[\frac{1-n}{1-\frac{m-1}{m+n-2}}+\frac{m-1}{\frac{m-1}{m+n-2}}\right]^2$$

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$$f''(x) = \frac{1}{\beta(m,n)}\left[x^{m-1}(1-x)^{n-1}\right]\left[\frac{-(n-1)}{\left(\frac{n-1}{m+n-2}\right)^2} + \frac{1-m}{\frac{(m-1)^2}{(m+n-2)^2}}\right.$$

$$\left. + \left[\frac{1-n}{\frac{m+n-2-m+1}{m+n-2}} + (m+n-2)\right]^2\right]$$

$$f''(x) = \frac{1}{\beta(m,n)}\left[x^{m-1}(1-x)^{n-1}\right]\left[\frac{-(n-1)(m+n-2)^2}{(n-1)^2} - \frac{m-1}{\frac{(m-1)^2}{(m+n-2)^2}}\right.$$

$$\left. + \left[\frac{-(n-1)(m+n-2)}{(n-1)} + (m+n-2)\right]^2\right]$$

$$f''(x) = \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\left[\frac{-(m+n-2)^2}{(n-1)} - \frac{(m+n-2)^2}{(m-1)}\right.$$

$$\left. + \left[-(m+n-2) + (m+n-2)\right]^2\right]$$

$$f''(x) = \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}(-1)(m+n-2)^2\left[\frac{1}{n-1} + \frac{1}{m-1}\right]$$

$$f''(x) = -\frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}(m+n-2)^2\frac{(m+n-2)}{(m-1)(n-1)}$$

$$f''(x) = -\frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\frac{(m+n-2)^3}{(m-1)(n-1)}$$

$$f''(x) < 0 \quad (\because m+n > 2,\ m,n > 1)$$

Hence $x = \frac{m-1}{m+n-2}$ is mode of Beta Distribution.

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# Q1(b) Show that for each Beta variate of Ist Kind, there corresponds a pair of Beta variate of 2nd Kind.

**Solution**:- As Beta variate of Ist kind is

$$\beta_1(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

let $x = \frac{1}{1+z} \Rightarrow z = \frac{1-x}{x}$

$dx = \frac{-1}{(1+z)^2}\cdot dz$  $\begin{cases} z \to \infty \text{ when } x \to 0 \\ z \to 0 \text{ when } x \to 1 \end{cases}$

Substituting these in $\beta_1$

$$\Rightarrow \int_\infty^0 \left[\frac{1}{1+z}\right]^{m-1}\left[1-\frac{1}{1+z}\right]^{n-1}\left[\frac{-dz}{(1+z)^2}\right]$$

$$= -\int_\infty^0 \left[\frac{1}{1+z}\right]^{m-1}\left[\frac{z}{1+z}\right]^{n-1}\left[\frac{dz}{(1+z)^2}\right]$$

$$= \int_0^\infty \frac{1}{(1+z)^{m-1}}\cdot\frac{z^{n-1}}{(1+z)^{n-1}}\cdot\frac{dz}{(1+z)^2}$$

$$= \int_0^\infty \frac{z^{n-1}}{(1+z)^{m-1+n-1+2}}\cdot dz$$

$$= \int_0^\infty \frac{z^{n-1}}{(1+z)^{m+n}}\cdot dz$$

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$$= \beta_2(m,n)$$

## Q8 (a) Show that chi-square distribution tends to normality as degree of freedom becomes very large

## Solution:-

See paper A/2014 Q8(b) at page "125"

A/2013

## Q8(b) Prove the reproductive property of $\chi^2$-distribution.

**Solution:-** If X and Y are independent $\chi^2$-random variables with $n_1$ & $n_2$ degree of freedom respectively, then the sum X+Y is a $\chi^2$-random variable with $n_1+n_2$ degree of freedom

Now $M_X(t) = (1-2t)^{-n_1/2}$

$M_Y(t) = (1-2t)^{-n_2/2}$

**Solved by Labeeb Ahmad**

let $W = X+Y$

Now the m-g-f on W is

$$M_W(t) = E[e^{tW}]$$
$$= E[e^{t(X+Y)}]$$

$$= E[e^{tX+tY}]$$

$$= \left(E[e^{tX}]\right)\cdot\left(E(e^{tY})\right)$$

$$= M_X(t) \cdot M_Y(t)$$

$$= (1-2t)^{-\frac{n_1}{2}} \cdot (1-2t)^{-\frac{n_2}{2}}$$

$$= (1-2t)^{-\frac{(n_1+n_2)}{2}}$$

This is m-g-f for $\chi^2$ random variable with $n_1+n_2$ degree of freedom.

Thus the sum of two independent chi-square is itself chi-square.

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**Q9(a)** Show that the Co-efficient of Co-relation is independent of origin & scale

**Solution:-** See Paper A|2015 Q8(a) at page "79"

A|2013

**Q(b)** Fit an equation of the form from the following data:

| X | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Y | 1.98 | 4.26 | 5.20 | 6.10 | 6.80 | 7.80 |

**Solution:-** The estimated regression line Y on X is $\hat{Y} = a + bX$

and the normal Equations are

$$\Sigma Y = na + b\Sigma X$$

$$\Sigma XY = a\Sigma X + b\Sigma X^2$$

To Compute the necessary sumation arrange the data in table.

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| | X | Y | XY | $X^2$ |
|---|---|---|---|---|
| | 1 | 1.98 | 1.98 | 1 |
| | 2 | 4.26 | 8.52 | 4 |
| | 3 | 5.20 | 15.6 | 9 |
| | 4 | 6.10 | 24.4 | 16 |
| | 5 | 6.80 | 34 | 25 |
| | 6 | 7.80 | 46.8 | 36 |
| $\Sigma$ | 21 | 32.14 | 131.3 | 91 |

$$\bar{X} = \frac{\Sigma X}{n} = \frac{21}{6} = 3.5$$

$$\bar{Y} = \frac{\Sigma Y}{n} = \frac{32.14}{6} = 5.35$$

$$b = \frac{n\Sigma XY - \Sigma X \Sigma Y}{n\Sigma X^2 - (\Sigma X)^2}$$

$$b = \frac{6(131.3) - 21(32.14)}{6(91) - (21)^2}$$

$$b = \frac{787.8 - 674.94}{546 - 441}$$

$$b = \frac{112.86}{105} = 1.074$$

And

$$a = \bar{Y} - b\bar{X}$$

$$a = 5.35 - (1.074)(3.5)$$

$$a = 5.35 - 3.759$$

$$a = 1.591$$

Hence $\hat{Y} = a + bX = 1.591 + 1.074X$

Solved by
Labeeb Ahmad

# PAPER A/2012

Solved by
Labeeb Ahmad

A/2012 , A/2007

## Q1(a) Describe the various approces to define probability

**Solution** :- There are three approaches to definding the Probability are:

(i) The classical Defination
(ii) The Relative Frequency
(iii) The axiomatic Defination

## Classical Defination

If a random experiment can produce "n" mutually exclusive and equally likly outcomes and if "m" out of these outcomes are considerd favorable to the occurence of a certain event A, then the

Probability of the event A, denoted by P(A), is defined as the ratio m/n. Symbolically, we write

$$P(A) = \frac{m}{n} = \frac{\text{No. of favorable outcome}}{\text{Total No. of possible outcomes}}$$

## (ii) The Relative Frequency

If a random experiment is repeated a large no. of times, say n, under identical conditions and if an event A is observed to occur m times. Then the probability of the event A is defined as the limit of the relative frequence $\frac{m}{n}$ as $n \to \infty$

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we write

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

## (iii) The Axiomatic Definition

This definition is based on a set of

axioms, where an axiom is a statment that is assumed to be true. Let $S$ be a sample space with sample points $E_1, E_2, E_3 --- E_i --- E_n$. To each sample point we assign a real number denoted by $P(E_i)$ and called the Probability of $E_i$ that must satisfy the following basic axioms

(i) For any event $E_i$ $0 \leq P(E_i) \leq 1$

(ii) $P(S)$ for the sure events $S$.

(iii) If $A$ and $B$ are mutually exclusive events. Then

$P(A \cup B) = P(A) + P(B)$

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A|2012

**Q1(b)** stat and Prove Baye's Theorem.

**Proof** See Paper A|2015 Q3(b) At Page "55"

A|2012

**Q1(c)** Bag I contains 3 white and 4 black balls while Bag II contains 5 white and 3 black balls. One ball is drawn from the Bag I and placed

unseen in Bag II What is the probability that a ball now drawn from Bag II is

(i) Black (ii) white

**Solution** :- Bag I 3 W , 4 B

Bag II 5 W , 3 B

Let A be the event that a ball is drawn from Bag I is following ways

$A_1$ = 1 white , $A_2$ = 1 black

$P(A_1) = {}^3C_1 \div {}^7C_2 = 1/7$

$P(A_2) = 4C_1 \div 7C_2 = 4/21$

The 2nd Bag after transforming 1 Ball

(i) 6W , 3B OR (ii) 5W , 4B

(i) Now Let B be the event that a black Ball is drawn. then

$P(B) = P(B \cap A_1) + P(B \cap A_2)$

$= P(A_1) P(B/A_1) + P(A_2) P(B/A_2)$

$= \frac{1}{7} \times \frac{3}{9} + \frac{4}{21} \times \frac{4}{9}$

$= \frac{3}{63} + \frac{16}{189} = \frac{25}{189}$

**Solved by Labeeb Ahmad**

(ii) If White ball is drawn

$P(W) = P(W \cap A_1) + P(W \cap A_2)$

$= P(A_1) \cdot P(W/A_1) + P(A_2) P(W/A_2)$

$= \frac{1}{7} \frac{6}{9} + \frac{4}{21} \times \frac{5}{9}$

$= \frac{6}{63} + \frac{20}{189} = \frac{38}{189}$

A/2012 Pg = 174

**Q2(a)** Prove that mean and variance of Binomial Distribution are $np$ and $np(1-p)$ respectively.

**Solution:-**

$$\text{Mean} = E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^{x} q^{n-x}$$

$$E(X) = \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} \cdot p^{x} \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n(n-1)!}{x(x-1)!\,(n-x)!} p^{x} \cdot q^{n-x}$$

$$= np \sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!\,[(n-1)-(x-1)]!} \cdot p^{x-1} \cdot q^{[(n-1)-(x-1)]!}$$

put $n-1 = m$

And $x-1 = y$

$$= np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^{y} q^{m-y}$$

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$$= np \sum_{y=0}^{m} \binom{m}{y} p^{y} q^{m-y}$$

$$= np\,(1) \quad \therefore \left( \sum_{y=0}^{m} \binom{m}{y} p^{y} q^{m-y} = 1 \right)$$

**Mean = $np$**

By Definition, variance $\sigma^2$ is given by $\sigma^2 = E(X-\mu)^2 = E(X^2) - [E(x)]^2$

$$\sigma^2 = E(X(X-1)+X) - [E(X)]^2$$

$$\sigma^2 = E(X(X-1)) + E(X) - [E(X)]^2 \quad \text{—— (i)}$$

$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)} \cdot \frac{n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!(n-x)!} \cdot p^x \cdot q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=0}^{n} \frac{(n-2)!}{(x-2)![(x-2)-(n-2)]!} \cdot p^{x-2} \cdot q^{[(x-2)-(n-2)]}$$

Put $n-2 = m$

And $x-2 = y$

$$= n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} \cdot p^y q^{m-y}$$

$$= n(n-1)p^2 \sum_{y=0}^{m} \binom{m}{y} p^y \cdot q^{m-y}$$

$$= n(n-1)p^2 \cdot (1)$$

$$= n(n-1)p^2$$

**Solved by Labeeb Ahmad**

Put in (i)

$$\sigma^2 = n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\boxed{\sigma^2 = np(1-p)}$$

A|2012 Pg=176

## Q2(b)

The Probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease what is the probability that

(i) At least 12 Survive

(ii) Exactly 5 Survive

**Solution:** Here $p = 0.4$ & $n = 15$

$$q = 1 - p = 1 - 0.4 = 0.6$$

(i) $$P(X \geq 12) = \sum_{x=12}^{15} {}^{15}C_x (0.4)^x (0.6)^{15-x}$$

$$= {}^{15}C_{12} (0.4)^{12} (0.6)^{15-12} + {}^{15}C_{13} (0.4)^{13} (0.6)^{15-13} + {}^{15}C_{14} (0.4)^{14} (0.6)^{15-14} + {}^{15}C_{15} (0.4)^{15} (0.6)^{15-15}$$

$$= 0.001648 + 0.00025 + 0.000024 + 0.0000107$$

$$= 0.001923$$

(ii) Exactly 5 survive

$$P(X = 5) = {}^{15}C_5 (0.4)^5 (0.6)^{15-5}$$

$$= 0.1860$$

**Solved by**
**Labeeb Ahmad**

A|2012 Pg = 177

# Q3(a) A Continuous r-v has the d-f

$$f(x) = \begin{cases} \frac{3}{4}(x)(2-x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$ using

(i) given density function $f(x)$

**Solved by Labeeb Ahmad**

## Solution :-

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$\mu = \int_0^2 \frac{3}{4}(x)(x)(2-x) \cdot dx$$

$$= \frac{3}{4} \int_0^2 (2x^2 - x^3) \cdot dx$$

$$= \frac{3}{4} \left[ 2 \cdot \frac{x^3}{3} - \frac{x^4}{4} \right|_0^2$$

$$= \frac{3}{4} \left[ \frac{2}{3}(2)^3 - \frac{(2)^4}{4} - 0 + 0 \right]$$

$$= \frac{3}{4} \left[ \frac{16}{3} - \frac{16}{4} \right]$$

$$= \frac{3}{4} \left[ \frac{64 - 48}{12} \right]$$

$$= \frac{3}{4}\left[\frac{16}{12}\right]$$

$$= \frac{3}{4}\left[\frac{4}{3}\right]$$

$$\mu = 1$$

**Solved by Labeeb Ahmad**

Now $E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx$

$$= \frac{3}{4}\int_0^2 x^2 \cdot x \cdot (2-x)\, dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\, dx$$

$$= \frac{3}{4}\left[2 \cdot \frac{x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{1}{2}(2)^4 - \frac{(2)^5}{5} - 0 + 0\right]$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right]$$

$$= \frac{3}{4}\left[\frac{40-32}{5}\right]$$

$$= \frac{3}{4}\left[\frac{8}{5}\right]$$

$$E(x^2) = \frac{6}{5}$$

$Var(X) = E(X^2) - [E(X)]^2$

$\sigma^2 = \frac{6}{5} - 1 = \frac{1}{5}$

$S\text{-}D = \sigma = \sqrt{1/5} = 0.44721$

$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$

$= P(1 - 2(0.44721) \leq X \leq 1 + 2(0.44721))$

$= P(0.1055 \leq X \leq 1.89442)$

$\Rightarrow \frac{3}{4} \int_{0.1055}^{1.89422} (x)(2-x).\,dx$

$= \frac{3}{4} \int_{0.1055}^{1.89422} (2x - x^2).\,dx$ **Solved by Labeeb Ahmad**

$= \frac{3}{4} \left\{ 2 \cdot \frac{x^2}{2} - \frac{x^3}{3} \right|_{0.1055}^{1.89422}$

$= \frac{3}{4} \left\{ \left[ (1.89422)^2 - \frac{(1.89422)^3}{3} \right] - \left[ (0.1055)^2 - \frac{(0.1055)^3}{3} \right] \right\}$

$= \frac{3}{4} \left[ 1.3225 - 0.01073 \right]$

$= \frac{3}{4} (1.31176)$

$= 0.98382$

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# Q3(b) The following table shows the Distribution function of random variable

| $x$ | $x \leq 1$ | $x \leq 2$ | $x \leq 3$ | $x \leq 4$ |
|---|---|---|---|---|
| $F(x)$ | 1/8 | 3/8 | 3/4 | 1 |

Find (i) Probability distribution of r-vX

(ii) $P(1 \leq x \leq 3)$

(iii) $P(x > 4)$

(iv) $P(x = 5)$

(v) $E(x)$

Solved by Labeeb Ahmad

**(i)**

| $x_i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x_i)$ | $\frac{1}{8}$ | $\frac{2}{8}$ | $3/8$ | $1/4$ |

**(ii)** $P(1 \leq x \leq 3) \Rightarrow$

If $F(x)$ is the distribution function and a & b be two real number such that $a < b$

Then probability of the interval is given by

$$P(a \leq x \leq b) = F(b) - F(a)$$

Thus $P(1 \le X \le 3) = F(3) - F(1)$

$= \frac{3}{4} - \frac{1}{8} = \frac{6-1}{8}$

$= 5/8$

**(iii)**

$P(X > 4) = 1 - P(X \le 4)$

$= 1 - F(4)$

$= 1 - 1$

$= 0$

**(iv)**

$P(X \le 5) = F(5) = 0$

**(v)** $E(X) = \sum_{i=1}^{\infty} x_i f(x_i)$

$= \sum_{i=1}^{4} x_i f(x_i)$

$= x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + x_4 f(x_4)$

$= 1 \cdot \left[\frac{1}{8}\right] + 2\left[\frac{2}{8}\right] + 3\left[\frac{3}{8}\right] + 4\left[\frac{1}{4}\right]$

$= \frac{1}{8} + \frac{4}{8} + \frac{9}{8} + 1$

$= \frac{1+4+9+8}{8}$

$= \frac{22}{8} = \frac{11}{4} = 2.75$

Solved by Labeeb Ahmad

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## Q4(a) Find the mean and variance of hypergeometric Probability distribution.

**Solution:** Let X have the hypergeometric probability distribution is given by

$$h(x;N;n;k) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \quad \text{for s.t } 0 \le x \le n \text{ and } 0 \le x \le k$$

## Mean

$$\mu = E(X)$$

$$\mu = \sum_{x=0}^{n} x \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{k} \frac{x\left\{\frac{k!}{x!(k-x)!}\right\}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{n} \frac{x\left\{\frac{k(k-1)!}{x(x-1)!(k-x)!}\right\}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=1}^{n} \frac{k\binom{k-1}{x-1}\binom{N-k}{n-x}}{\binom{N}{n}}$$

Solved by
Labeeb Ahmad

put $y = x-1$

So $y = 0$ when $x=1$ and $y = n-1$ when $x = n$

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$$y = k \frac{\sum_{j=0}^{n-1} \binom{k-1}{y}\binom{N-k}{n-1-y}}{\binom{N}{n}}$$

As we know

$$\therefore \quad \boxed{\sum_{j=0}^{m} \binom{a}{j}\binom{b}{m-j} = \binom{a+b}{m}}$$

$$y = k \frac{\binom{k-1+N-k}{y+n-1-y}}{\binom{N}{n}}$$

$$= \frac{k \binom{N-1}{n-1}}{\binom{N}{n}}$$

$$= \frac{k \left[ \dfrac{(N-1)!}{(n-1)!(N-1-n+1)!} \right]}{\dfrac{N!}{n!(N-n)!}}$$

$$= \frac{k (N-1)!}{(n-1)! (N-n)!} \times \frac{n! (N-n)!}{N!}$$

$$= \frac{k (N-1)!}{(n-1)!} \times \frac{n (n-1)!}{N (N-1)!}$$

$$= \frac{kn}{N}$$

$$y = np \quad \text{where } p = \frac{k}{N}$$

Solved by
Labeeb Ahmad

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# Variance

$$= \sigma^2 = E(X^2) - [E(X)]^2$$

$$= E[X(X-1)+X] - [E(X)]^2$$

$$= E[X(X-1)] + E(X) - E(X)^2 \quad \text{---} \; (1)$$

First Solve

$$E[X(X+1)] = \sum_{x=0}^{n} x(x+1) \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{\left\{\frac{k!}{x!(k-x)!}\right\}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)} \frac{\left\{\frac{k(k-1)(k-2)!}{\cancel{x(x-1)}(x-2)!(k-x)!}\right\}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=2}^{n} \frac{\left\{\frac{k(k-1)(k-2)!}{(x-2)!(k-x)!}\right\}\binom{N-k}{n-x}}{\binom{N}{n}}$$

Put $y = x-2$

$y = 0$ when $x = 2$ & $y = n-2$ when $x = n$

$$= k(k-1) \sum_{y=0}^{n-2} \frac{\frac{(k-2)!}{y!(k-y)!}\binom{N-k}{n-2-y}}{\binom{N}{n}}$$

Solved by
Labeeh Ahmad

$$E[(x)(x-1)] = \frac{k(k-1)\sum_{y=0}^{n-2}\binom{k-2}{y}\binom{N-k}{n-2-y}}{\binom{N}{n}}$$

using $\sum_{j=0}^{m}\binom{a}{j}\binom{b}{m-j} = \binom{a+b}{m}$

$$= k(k-1)\binom{k-2+N-k}{y+n-2-y} \Big/ \binom{N}{n}$$

$$= k(k-1)\binom{N-2}{n-2} \Big/ \binom{N}{n}$$

$$= k(k-1)\frac{(N-2)!}{(n-2)!\,(N-n)!} \times \frac{n!\,(N-n)!}{N!}$$

$$= k(k-1)\frac{(N-2)!}{(n-2)!} \times \frac{n(n-1)(n-2)!}{N(N-1)(N-2)!}$$

$$E[x(x-1)] = \frac{k(k-1)\,n(n-1)}{N\cdot(N-1)}$$

As $\sigma^2 = E[x(x-1)] + E(x) - E(x)^2$

$$= \frac{k(k-1)n(n-1)}{N(N-1)} + \frac{kn}{N} - \frac{k^2n^2}{N^2}$$

$$= \frac{kn}{N}\left[\frac{(k-1)(n-1)}{N-1} + 1 - \frac{kn}{N}\right]$$

$$= \frac{kn}{N}\left[\frac{N(k-1)(n-1) + N(N-1) - (N-1)(kn)}{N(N-1)}\right]$$

Solved by
Labeeb Ahmad

As $\frac{R}{N} = P$

$$\sigma^2 = nP\left[\frac{N(Rn - R - n + 1) + N^2 - N - NRn + nR}{N(N-1)}\right]$$

$$\sigma^2 = \frac{nP}{N(N-1)}\left[\cancel{NRn} - NR - nN + \cancel{N} + N^2 - \cancel{N} - \cancel{NRn} + nR\right]$$

$$= \frac{nP}{N(N-1)}\left[N^2 - Nn - NR + Rn\right]$$

$$= \frac{nP}{N(N-1)}\left[N(N-n) - R(N-n)\right]$$

$$= \frac{nP}{N(N-1)}\left((N-R)(N-n)\right)$$

$$= nP \cdot \frac{(N-R)}{N} \cdot \frac{(N-n)}{(N-1)}$$

$$\sigma^2 = nPq \cdot \frac{N-n}{N-1} \quad \text{where } q = \frac{N-R}{N}$$

Solved by
Labeeb Ahmad

A/2012 , A/2010



**Q4(b)** Derive moment generating function of Possion Distribution. And using m-g-f, show Mean and variance of Possion Distribution is equal

**Solution:-** By Definition $M_o(t) = E(e^{tx})$

$$M_o(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\mu} \mu^x}{x!}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(e^t \mu)^x}{x!}$$

$$= e^{-\mu} \left[ 1 + \frac{(e^t\mu)^1}{1!} + \frac{(e^t\mu)^2}{2!} + \text{———} \right]$$

$$= e^{-\mu} \left[ e^{(e^t \mu)} \right]$$

**Solved by**
**Labeeb Ahmad**

$$M_o(t) = e^{\mu[e^t - 1]}$$

Now we prove Mean and variance are equal using m-g-f

$$\mu_1' = \lim_{t \to 0} \left[ \frac{d M_o(t)}{dt} \right] = \lim_{t \to 0} \frac{d}{dt} \left[ e^{\mu(e^t - 1)} \right]$$

$$= \lim_{t \to 0} \left[ e^{\mu(e^t - 1)} \mu . e^t \right]$$

$\mu_1' = e^{\mu[e^0-1]} \cdot \mu e^0$

$= e^{\mu(1-1)} \cdot \mu e^0$

$= e^0 \cdot \mu \cdot 1$

$= \mu$

**Solved by Labeeb Ahmad**

$\boxed{\text{Mean} = \mu}$

Now

$\mu_2' = \lim_{t\to 0} \frac{d^2}{dt^2} M_0(t)$

$= \lim_{t\to 0} \frac{d}{dt}\left[\frac{d}{dt}\left(e^{\mu(e^t-1)}\right)\right]$

$= \lim_{t\to 0} \frac{d}{dt}\left[e^{\mu(e^t-1)} \cdot \mu e^t\right]$

$= \mu \lim_{t\to 0} \frac{d}{dt}\left(e^{\mu(e^t-1)} \cdot e^t\right)$

$= \mu \lim_{t\to 0} \frac{d}{dt}\left[e^{\mu e^t-\mu+t}\right]$

$= \mu \lim_{t\to 0} e^{(\mu e^t-\mu+t)} (\mu e^t - 0 + 1)$

$= \mu\left[e^{\mu e^0-\mu+0} (\mu e^0+1)\right]$

$= \mu\left[e^0 (\mu+1)\right]$

$\mu_2' = \mu^2+\mu$

Variance $= \mu_2' - \mu_1'^2 = \mu^2 + \mu - \mu^2 = \mu$

Hence Mean = Variance

A/2012



**Q5(a)** Derive the m.g.f of $X \sim (\mu, \sigma^2)$. Use m.g.f of r.v $X$ to deduce the m.g.f of standard normal variate i-e $Z \sim (0,1)$

**Solution:** Let the r.v $X$ be $N(\mu, \sigma^2)$ then the m.g.f of $X$ w.r.t the origin is given by

Solved by
Labeeb Ahmad

$$M_o(t) = E(e^{tx}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \, dx$$

put $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$

$dx = \sigma dz$

$\begin{cases} z \to \infty & \text{when } x \to \infty \\ z \to -\infty & \text{when } x \to -\infty \end{cases}$

$$M_o(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \, e^{-z^2/2} \, \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu + t\sigma z - \frac{z^2}{2}} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z^2 - 2t\sigma z]} \, dz$$

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$$M_o(t) = \frac{1}{\sqrt{2\pi}} e^{4t} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z^2 - 2t\sigma z + \sigma^2 t^2 - \sigma^2 t^2]} \cdot dz$$

$$= \frac{e^{4t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-\sigma t)^2 - \sigma^2 t^2]} \cdot dz$$

$$= \frac{e^{4t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} \cdot dz \quad \text{——— (1)}$$

Take $\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$

$$= 2 \int_{0}^{\infty} e^{-\frac{1}{2}[z-\sigma t]^2} \cdot dz \quad \text{——— (2)}$$

put $\frac{1}{2}(z - \sigma t)^2 = \theta$

**Solved by Labeeb Ahmad**

$$z - \sigma t = \sqrt{2\theta}$$

$$z = \sqrt{2\theta} + \sigma t$$

$$dz = \sqrt{2} \cdot \frac{1}{2\sqrt{\theta}} \cdot d\theta + 0$$

$$dz = \frac{d\theta}{\sqrt{2\theta}}$$

$\theta \to \infty$ when $z \to \infty$

$\theta \to 0$ when $z \to 0$

Putting these in (2)

$$= 2 \int_{0}^{\infty} e^{-\theta} \frac{d\theta}{\sqrt{2\theta}}$$

$$= \frac{2}{\sqrt{2}} \int_0^\infty e^{-\theta} \theta^{-1/2} . d\theta$$

$$= \sqrt{2} \int_0^\infty e^{-\theta} \theta^{\frac{1}{2}-1} d\theta$$

$$= \sqrt{2}\,\Gamma(1/2)$$

$$= \sqrt{2}\,\sqrt{\pi} \quad \because \Gamma(1/2) = \sqrt{\pi}$$

$$= \sqrt{2\pi}$$

Put in (1)

$$M_o(t) = \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{1}{2}\sigma^2 t^2} . \sqrt{2\pi}$$

$$M_o(t) = e^{\mu t + \sigma^2 t^2} \quad \text{—— (3)}$$

In standard Normal variate

i-e $Z \sim N(0,1)$

Mean $= \mu = 0$

Variance $= \sigma^2 = 1$

Solved by Labeeb Ahmad

Put in (3)

$$M_o(t) = e^{0t + \frac{1}{2}t^2}$$

$$= e^{t^2/2}$$

that is m.g.f of standard Normal variate.

A/2012 

**Q5(b)** In a Mathematics examination the grades are normally distributed with mean of 80 and standard deviation of 8. If X is the r.v showing the grades of students in this exam then find

(i) $P(80 \leq X \leq 95)$

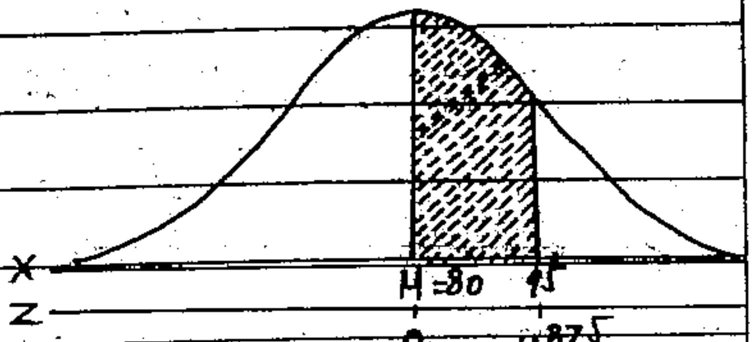
(ii) $P(X \leq 60)$

**Solution:-** We draw the normal curve sketch showing x and z values, and the desired area for each part with $\mu = 80$ and $\sigma = 8$ we have

(i) $P(80 \leq x \leq 95)$

at $x = 80$

$z = \frac{80-80}{8}$




$z = 0$

and at $x = 95$

$z = \frac{95-80}{8} = 1.875$

Solved by Labeeb Ahmad

$P(80 \leq x \leq 95) = P(0 \leq z \leq 1.875)$

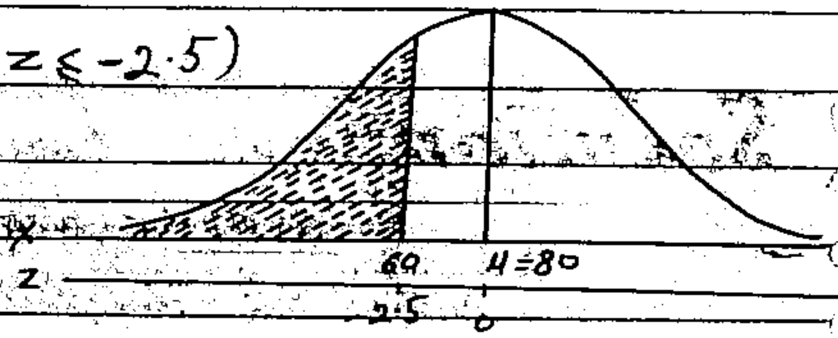
Now using the table (Area under the unit normal curve)

$P(0 \leq Z \leq 1.875) = 0.4693$

**(ii)** $P(X \leq 60)$

at $x = 60$, $Z = \frac{60-80}{8} = -2.5$

$P(X \leq 60) = P(Z \leq -2.5)$




$P(Z \leq -2.5) = 0.5 - P(-2.5 \leq Z < 0)$

using Table

$= 0.5 - 0.4938$

$= 0.0062$

Solved by
Labeeb Ahmad

A/2012, A/2008 pg=194

## Q6(a)

Find the mean and variance and moments generating function of an exponential distribution. Show that its mean and standard deviation are equal.

## Solution:-

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Mean $= E(x)$

$$= \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \int_{0}^{\infty} x \lambda e^{-\lambda x} \cdot dx$$

$$= \lambda \int_{0}^{\infty} x \cdot e^{-\lambda x} \cdot dx \quad \text{—— (i)}$$

put $\lambda x = t$ or $x = \frac{t}{\lambda}$

$\lambda \cdot dx = dt$ $\quad dx = \frac{dt}{\lambda}$

$t \to 0$ when $x \to 0$ and $t \to \infty$ when $x \to \infty$

Putting these in (i)

$$= \lambda \int_{0}^{\infty} \frac{t}{\lambda} e^{-t} \cdot \frac{dt}{\lambda}$$

$$= \frac{1}{\lambda} \int_{0}^{\infty} t^{2-1} e^{-t} \cdot dt$$

Solved by
Labeeb Ahmad

As $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx$ pg=195

$\mu = \frac{1}{\lambda} \Gamma(2)$

$\mu = \frac{1}{\lambda}$ —— (ii)

Now $E(x^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} \, dx$

$= \lambda \int_0^\infty x^2 e^{-\lambda x} \, dx$

put $\lambda x = t$ | $t \to 0$ when $x \to 0$

$x = t/\lambda$ | $t \to \infty$ when $x \to \infty$

$dx = dt/\lambda$

Solved by
Labeeb Ahmad

$E(x^2) = \lambda \int_0^\infty \left(\frac{t}{\lambda}\right)^2 e^{-t} \frac{dt}{\lambda}$

$= \frac{1}{\lambda^2} \int_0^\infty t^2 e^{-t} \, dt$

$= \frac{1}{\lambda^2} \int_0^\infty t^{3-1} e^{-t} \, dt$

$= \frac{1}{\lambda^2} \Gamma(3) = \frac{1}{\lambda^2} 2! = \frac{2}{\lambda^2}$

Variance = $E(x^2) - [E(x)]^2$

$$\sigma^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$\sigma^2 = \frac{1}{\lambda^2}$$

$S\text{-}D = \sigma = 1/\lambda$ —— (iii)

From (ii) & (iii) It is clear

Mean = S-D

Now we find Moment generating function

$$M_0(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, dx$$

Solved by Labeeb Ahmad

$$= \lambda \int_0^\infty e^{-\lambda x + tx} \, dx$$

$$= \lambda \int_0^\infty e^{-x(\lambda - t)} \, dt$$

$$= \lambda \left[ \frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty$$

$$= \frac{-\lambda}{\lambda - t} \left[ e^{-\infty} - e^{-0} \right] = \frac{-\lambda}{\lambda - t} [0 - 1]$$

$$M_0(t) = \frac{\lambda}{\lambda - t} \quad \text{for } \lambda > t$$

A/2012

# Q6(b)

Stat and Poove Central Limit Theorem

## Solution:

See paper A/2016

Q5(b) at Page "21"

A/2012 Pg = 198

## Q7(a) The following data are the selling prices y, of a certain make and model of used car x years old

| x | 1 | 2 | 2 | 3 | 5 | 5 |
|---|---|---|---|---|---|---|
| y | 2350 | 1695 | 1750 | 1395 | 985 | 895 |

(i) Fit an exponential curve of the form $\hat{y} = ab^x$

(ii) Estimate the selling price of such a car when it is 4 years old.

## Solution:-

(i) Taking a regression of the form

$$\hat{y} = ab^x$$

Taking log both side

$$\log \hat{y} = \log(ab^x)$$

$$= \log a + \log b^x$$

**Solved by Labeeb Ahmad**

$$\log \hat{y} = \log a + x \log b$$

$$Y = A + xB$$

Now the two normal equations of regression lines are

$\Sigma Y = nA + B\Sigma X$ —— (1)

$\Sigma xY = A\Sigma x + B\Sigma x^2$ —— (2)

| $x$ | $y$ | $Y = \log y$ | $XY$ | $X^2$ |
|---|---|---|---|---|
| 1 | 2350 | 3.3710 | 3.3710 | 1 |
| 2 | 1695 | 3.2292 | 6.4584 | 4 |
| 2 | 1750 | 3.2430 | 6.486 | 4 |
| 3 | 1395 | 3.1446 | 9.4338 | 9 |
| 5 | 985 | 2.9934 | 14.967 | 25 |
| 5 | 895 | 2.9518 | 14.759 | 25 |
| 18 | | 18.9331 | 55.4753 | 68 |

Now putting the values in (1) & (2)

$18.9331 = 6A + 18B$ —— (3)

$55.4753 = 18A + 68B$ —— (4)

Equation (4) divided by "3"

$18.4917 = 6A + 22.667B$ —— (5)

Equation (3) subtracting from (3)

Solved by
Labeeb Ahmad

$18.9331 = 6A + 18B$

$\underline{-}18.4917 = \underline{-}6A \underline{\pm}22.667B$

$0.4414 = -4.667B$

$$B = \frac{0.4414}{-4.667} = -0.0945$$

$$\boxed{B = -0.0945}$$

Now putting this value in (3)

$$18.9331 = 6A + 18(-0.0945)$$

$$18.9331 = 6A - 1.701$$

$$18.9331 + 1.701 = 6A$$

$$6A = 20.6341$$

$$A = (20.6341)/6$$

$$\boxed{A = 3.439}$$

As $\log a = A$

$$a = \text{Anti Log } (A)$$

$$a = \text{Anti Log } (3.439)$$

$$\boxed{a = 2747.9}$$

**Solved by Labeeb Ahmad**

And $\log b = B$

$$b = \text{Anti Log } (B)$$

$$b = \text{Anti Log}(-0.0946)$$

$$\boxed{b = 0.804}$$

So the required equation is

$$\hat{y} = Ab^x$$

$$\hat{y} = 2747.9\,(0.804)^x$$

## (ii)

Put $x = 4$ in the least square regression line

$$\hat{y} = 2747.9\,(0.804)^4$$

$$\hat{y} = (2747.9)\,(0.4178)$$

$$\hat{y} = 1148.22$$

**Solved by**
**Labeeb Ahmad**

A|2012, A/2008, A/2004 Pg = 202

## Q7(b)

If $x_1, x_2, x_3$ are not correlated. Also $x_1, x_2, x_3$ have zero mean and equal variance $\sigma^2$, then show that Correlation between $(x_1+x_2)$ and $(x_1+x_3)$ is 1/2.

## Solution :-

Let $u = x_1 + x_2$

and $v = x_1 + x_3$

Then $r_{uv} = \dfrac{\sum(u-\bar{u})(v-\bar{v})}{\sqrt{\sum(u-\bar{u})^2 \cdot \sum(v-\bar{v})^2}}$ —— (1)

Solved by
Labeeb Ahmad

$u - \bar{u} = (x_1 + x_2) - (\bar{x}_1 + \bar{x}_2)$

$= (x_1 - \bar{x}_1) + (x_2 - \bar{x}_2)$

$\sum(u-\bar{u})^2 = \sum\left[(x_1-\bar{x}_1) + (x_2-\bar{x}_2)\right]^2$

$= \sum\left[(x_1-\bar{x}_1)^2 + (x_2-\bar{x}_2)^2 + 2(x_1-\bar{x}_1)(x_2-\bar{x}_2)\right]$

$= \sum(x_1-\bar{x}_1)^2 + \sum(x_2-\bar{x}_2)^2 + 2\sum(x_1-\bar{x}_1)(x_2-\bar{x}_2)$

$= n\sigma^2 + n\sigma^2 + 0$ $\quad\because x_1 \,\&\, x_2$ are not correlated

$\sum(u-\bar{u})^2 = 2n\sigma^2$

And $v - \bar{v} = (x_1 + x_3) - (\bar{x}_1 + \bar{x}_3)$

$= (x_1 - \bar{x}_1) + (x_3 - \bar{x}_3)$

$\sum(v-\bar{v})^2 = \sum\left[(x_1-\bar{x}_1) + (x_3-\bar{x}_3)\right]^2$

$$\sum(v-\bar{v})^2 = \sum\left[(x_1-\bar{x}_1)^2 + (x_3-\bar{x}_3)^2 + 2(x_1-\bar{x}_1)(x_3-\bar{x}_3)\right]$$

$$= \sum(x_1-\bar{x}_1)^2 + \sum(x_3-\bar{x}_3)^2 + 2\sum(x_1-\bar{x}_1)(x_3-\bar{x}_3)$$

$$= n\sigma^2 + n\sigma^2 + 0$$

$$\sum(v-\bar{v})^2 = 2n\sigma^2$$

And $\sum(u-\bar{u})(v-\bar{v}) = \sum\left(\left[(x_1-\bar{x}_1)+(x_2-\bar{x}_2)\right]\left[(x_1-\bar{x}_1)+(x_3-\bar{x}_3)\right]\right)$

$$= \sum(x_1-\bar{x}_1)^2 + \sum(x_1-\bar{x}_1)(x_3-\bar{x}_3) + \sum(x_2-\bar{x}_1)(x_1-\bar{x}_1) + \sum(x_2-\bar{x}_2)(x_3-\bar{x}_3)$$

$$= n\sigma^2 + 0 + 0 + 0$$

$$\sum(u-\bar{u})(v-\bar{v}) = n\sigma^2$$

Substituting these in (1), we get

$$r_{uv} = \frac{n\sigma^2}{\sqrt{(2n\sigma^2)(2n\sigma^2)}}$$

$$= \frac{n\sigma^2}{2n\sigma^2}$$

$$r_{uv} = 1/2$$

**Solved by**
**Labeeb Ahmad**

A/2012, A/2008 Pg = 204

## Q8(a)

If $x$ is $t$-distributed random variables with $n$ degree of freedom then show $\left(1+\frac{x^2}{n}\right)^{-1}$ has Beta Distribution.

## Solution :-

As

$$df(t) = \frac{1}{\sqrt{n}\;\beta\left(\frac{n}{2},\frac{1}{2}\right)}\left[1+\frac{t^2}{n}\right]^{-\frac{1}{2}[n+1]} dt$$

Let $w = \left(1+\frac{x^2}{n}\right)^{-1}$

replace $x$ by "$t$"

$$\text{OR} \Rightarrow w = \left(1+\frac{t^2}{n}\right)^{-1}$$

$$w^{-1} = 1+\frac{t^2}{n}$$

Solved by Labeeb Ahmad

$$n(w^{-1}-1) = t^2$$

$$t = \sqrt{n(w^{-1}-1)}$$

Differentiate both side

$$dt = \sqrt{n}\,\frac{1}{2\sqrt{w^{-1}-1}}\,(-w^{-2})\,.\,dw$$

Putting these in $df(t)$

Solved by Labeeb Ahmad

$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)} w^{\frac{n+1}{2}} \sqrt{n} \frac{1}{2\sqrt{w^{-1}-1}} (-w^{-2})\,dw$$

$$= \frac{-w^{\frac{n}{2}+\frac{1}{2}-2}}{2\,\beta\left(\frac{n}{2},\frac{1}{2}\right)\sqrt{\frac{1-w}{w}}}\,dw$$

$$= \frac{-w^{\frac{n}{2}-\frac{3}{2}}}{2\,\beta\left(\frac{n}{2},\frac{1}{2}\right)\frac{(1-w)^{1/2}}{w^{1/2}}}\,dw$$

$$= -\frac{1}{2}\left[\frac{1}{\beta\left(\frac{n}{2},\frac{1}{2}\right)} w^{\frac{n}{2}-\frac{3}{2}+\frac{1}{2}} (1-w)^{-1/2}\,dw\right]$$

$$= -\frac{1}{2}\left[\frac{1}{\beta\left(\frac{n}{2},\frac{1}{2}\right)} w^{\frac{n}{2}-1} (1-w)^{\frac{1}{2}-1} \cdot dw\right]$$

Put $\frac{n}{2} = n'$ and $\frac{1}{2} = m'$

$$= -\frac{1}{2}\left[\frac{1}{\beta(n',m')} \cdot w^{n'-1} (1-w)^{m'-1} \cdot dw\right]$$

This follows Beta Distribution.

**Solved by**
**Labeeb Ahmad**

A/2012, S/2012, S/2009, A/2008, A/2007, A/2006

**Q8(b)** The rth moment about origin $F \sim F_{v_1, v_2}$ is given by

$$\mu_r' = \frac{\left(\frac{v_2}{v_1}\right)^r \Gamma\left(\frac{v_1}{2}+r\right)\Gamma\left(\frac{v_2}{2}-r\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(v_2/2\right)}$$

**Solution :-** By defination

$$\mu_r' = E(F^r) = \int_0^\infty F^r \cdot f(F) \cdot dF$$

$$= \frac{\left(\frac{v_1}{v_2}\right)^{v_1/2}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^\infty \frac{F^r \, F^{\frac{v_1}{2}-1}}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_1+v_2}{2}}} dF \quad (1)$$

$\longrightarrow I_1$

Take $I_1$ & Put $\frac{v_1}{v_2}F = \theta$

$$\frac{v_1}{v_2} dF = d\theta \quad \text{or } dF = \frac{v_2}{v_1} d\theta$$

$$\begin{cases} \theta \to 0 & \text{when } F \to 0 \\ \theta \to \infty & \text{when } F \to \infty \end{cases}$$

Solved by
Labeeb Ahmad

1

pg = 207

$$I_1 = \int_0^{\infty} \frac{F^{\frac{v_1}{2}+r-1}}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_1+v_2}{2}}}\, dF$$

$$I_1 = \int_0^{\infty} \frac{\left(\frac{v_2}{v_1}\theta\right)^{\frac{v_1}{2}+r-1}}{(1+\theta)^{\frac{v_1+v_2}{2}}} \left(\frac{v_2}{v_1}\right) d\theta$$

$$I_1 = \int_0^{\infty} \frac{\left(\frac{v_2}{v_1}\right)^{\frac{v_1}{2}+r-1} \theta^{\frac{v_1}{2}+r-1}}{(1+\theta)^{(v_1+v_2)/2}} \cdot \frac{v_2}{v_1}\, d\theta$$

$$I_1 = \left(\frac{v_2}{v_1}\right)^{\frac{v_1}{2}+r} \int_0^{\infty} \frac{\theta^{\left(\frac{v_1}{2}+r\right)-1}}{(1+\theta)^{\left(\frac{v_1}{2}+r\right)+\left(\frac{v_2}{2}-r\right)}}\, d\theta$$

$$I_1 = \left(\frac{v_2}{v_1}\right)^{\frac{v_1}{2}+r} \beta\left(\frac{v_1}{2}+r, \frac{v_2}{2}-r\right)$$

**Solved by**
**Labeeb Ahmad**

$(I_1)$ put in (1)

$$\mu_r' = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} \left(\frac{v_2}{v_1}\right)^{\frac{v_1}{2}+r} \beta\left(\frac{v_1}{2}+r, \frac{v_2}{2}-r\right)}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)}$$

$$= \left(\frac{v_2}{v_1}\right)^{r} \frac{\Gamma\left(\frac{v_1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \cdot \frac{\Gamma\left(\frac{v_1}{2}+r\right)\Gamma\left(\frac{v_2}{2}-r\right)}{\Gamma\left(\frac{v_2}{2}+\frac{v_1}{2}\right)}$$

pg = 208

$$\mu_r' = \frac{(v_2/v_1)^r \, \Gamma\left(\frac{v_1}{2}+r\right) \Gamma\left(\frac{v_2}{2}-r\right)}{\Gamma\left(\frac{v_1}{2}\right) \cdot \Gamma\left(\frac{v_2}{2}\right)}$$

For mean put $r = 1$

$$\mu_1' = \frac{[v_2/v_1] \, \Gamma\left(\frac{v_1}{2}+1\right) \Gamma\left(\frac{v_2}{2}-1\right)}{\Gamma\left(\frac{v_1}{2}\right) \cdot \Gamma\left(\frac{v_2}{2}\right)} \quad \text{--- (2)}$$

As $\Gamma\left(\frac{v_1}{2}+1\right) = \frac{v_1}{2} \Gamma\left(\frac{v_1}{2}\right)$

And $\Gamma\left(\frac{v_2}{2}-1\right) = \frac{\left(\frac{v_2}{2}-1\right)}{\left(\frac{v_2}{2}-1\right)} \Gamma\left(\frac{v_2}{2}-1\right)$

$$= \frac{\Gamma(v_2/2)}{\left(\frac{v_2}{2}-1\right)}$$

Solved by
Labeeb Ahmad

Putting these in (2)

$$\mu_1' = \frac{(v_2/v_1)\left[\frac{v_1}{2}\right] \Gamma\left(\frac{v_1}{2}\right) \cdot \Gamma\left(\frac{v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right) \cdot \Gamma\left(\frac{v_2}{2}\right) \left(\frac{v_2}{2}-1\right)}$$

$$\mu_1' = \frac{v_2/2}{\frac{v_2-2}{2}} = \frac{v_2}{v_2-2} \quad \text{— Mean}$$

Now put $r=2$ for variance

$$\mu_2' = \frac{\left(\frac{v_2}{v_1}\right)^2 \Gamma\left(\frac{v_1}{2}+2\right)\Gamma\left(\frac{v_2}{2}-2\right)}{\Gamma\left(\frac{v_1}{2}\right)\cdot\Gamma\left(\frac{v_2}{2}\right)} \quad (3)$$

$$\Gamma\left(\frac{v_1}{2}+2\right) = \left(\frac{v_1}{2}+1\right)\left(\frac{v_1}{2}\right)\Gamma\left(v_1/2\right)$$

And

$$\Gamma\left(\frac{v_2}{2}-2\right) = \frac{\left(\frac{v_2}{2}-1\right)\left(\frac{v_2}{2}-2\right)\Gamma\left(\frac{v_2}{2}-2\right)}{\left(\frac{v_2}{2}-1\right)\left(\frac{v_2}{2}-2\right)}$$

$$= \frac{\Gamma\left(\frac{v_2}{2}\right)}{\left(\frac{v_2}{2}-1\right)\left(\frac{v_2}{2}-2\right)}$$

Solved by
Labeeb Ahmad

Putting these in equ (3)

$$\mu_2' = \frac{\left(\frac{v_2}{v_1}\right)^2\left(\frac{v_1}{2}+1\right)\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_1}{2}\right)\quad\Gamma\left(\frac{v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)\quad\left(\frac{v_2}{2}-1\right)\left(\frac{v_2}{2}-2\right)}$$

$$\mu_2' = \frac{\frac{v_2^2}{v_1^2}\left[\frac{v_1+2}{2}\right]\left[\frac{v_1}{2}\right]}{\left[\frac{v_2-2}{2}\right]\left[\frac{v_2-4}{2}\right]}$$

Solved by
Labeeb Ahmad

$$\mu_2' = \frac{\frac{v_2^2 (v_1+2)}{v_1 \cancel{4}}}{\frac{(v_2-2)(v_2-4)}{\cancel{4}}}$$

$$\mu_2' = \frac{v_2^2 (v_1+2)}{v_1 (v_2-2)(v_2-4)}$$

**Variance** $= \mu_2' - \mu_1'^2$

$$\sigma^2 = \frac{v_2^2 (v_1+2)}{v_1 (v_2-2)(v_2-4)} - \frac{v_2^2}{(v_2-2)^2}$$

$$\sigma^2 = \frac{v_2^2}{(v_2-2)} \left[ \frac{(v_1+2)}{v_1(v_2-4)} - \frac{1}{(v_2-2)} \right]$$

$$\sigma^2 = \frac{v_2^2}{(v_2-2)} \left[ \frac{(v_1+2)(v_2-2) - v_1(v_2-4)}{v_1(v_2-4)(v_2-2)} \right]$$

$$\sigma^2 = \frac{v_2^2}{(v_2-2)} \left[ \frac{\cancel{v_1 v_2} - 2v_1 + 2v_2 - 4 - \cancel{v_1 v_2} + 4v_1}{v_1(v_2-4)(v_2-2)} \right]$$

$$\sigma^2 = \frac{v_2^2 (2v_2 + 2v_1 - 4)}{v_1 (v_2-4)(v_2-2)^2}$$

Solved by
Labeeb Ahmad

A/2012

Pg = 211

**Q9(a)** Derive the chi-square distribution as limiting form of F-distribution.

**Solution:** See Paper A/2016 Q8(a) at page "33"

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A/2012

**Q9(b)** If X and Y are two independent random variable such that $X \sim \chi^2_{n_1}$ and $Y \sim \chi^2_{n_2}$ Then $X + Y \sim \chi^2_{n_1+n_2}$

**Solution:-**

As $X \sim \chi^2_{n_1}$

$\Rightarrow M_X(t) = (1-2t)^{-n_1/2}$

and

Solved by
Labeeb Ahmad

$Y \sim \chi^2_{n_2}$

$\Rightarrow M_Y(t) = (1-2t)^{-n_2/2}$

let $W = X + Y$

Now the m.g.f of

$M_W(t) = E[e^{tW}]$

$$M_W(t) = E[e^{tW}]$$
$$= E[e^{t(X+Y)}]$$
$$= E[e^{tX+tY}]$$
$$= E[e^{tX} \cdot e^{tY}]$$
$$= E[e^{tX}] \cdot E[e^{tY}]$$
$$= M_X(t) \cdot M_Y(t)$$
$$= (1-2t)^{-\frac{n_1}{2}} \cdot (1-2t)^{-\frac{n_2}{2}}$$
$$= (1-2t)^{-\left[\frac{n_1+n_2}{2}\right]}$$

This is a m-g-f of $\chi^2$ - r.v with $n_1+n_2$ degree of freedom

Thus $X+Y \sim \chi^2_{n_1+n_2}$

**Solved by**
**Labeeb Ahmad**

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# PAPER A/2011

A/2011

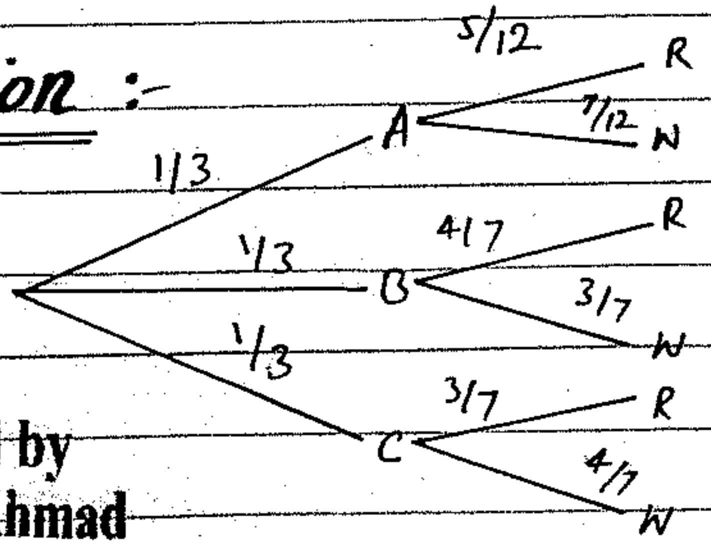
## Q1(a)

Description of the three urns of the same appearance are given as under Urn A contains 5 red and 7 white balls. Urn B contains 4 red and 3 white balls. Urn C contains 3 red and 4 white balls. An urn is selected at random and a ball is drawn from the urn.

(i) What is the probability that the ball drawn is red?

(ii) If the ball is red drawn, what is the probability that it came from urn A.

## Solution :-




Solved by
Labeeb Ahmad

Here we first select one of three urns and then we draw a ball which is either red (R) or white (W). In other words, we perform a sequence of two experiments.

This process is described by the probability tree diagram in privious page. In which each branch of the tree gives the respective probability.

Now the probability of selecting Urn A, for instance and then a red is $\frac{1}{3} \cdot \frac{5}{12} = \frac{5}{36}$

Solved by
Labeeb Ahmad

Because the probability that any particular path of the tree occur is by the multiplication law. The product of the probability of each branch of the path.

Solved by
Labeeb Ahmad

(i) Now the probability of drawing a red ball is given by the relation

$$P(R) = P(A)\,P(R/A) + P(B)\cdot P(R/B) + P(C)\;P(R/C)$$

as three mutually exclusive paths leading to the drawing of red ball

Hence $P(R) = \frac{1}{3}\cdot\frac{5}{12} + \frac{1}{3}\cdot\frac{4}{7} + \frac{1}{3}\cdot\frac{3}{7}$

$= \frac{119}{252} = 0.4722$

(ii)

Here we need the probability that Urn A is selected, given that the ball is drawn red that is

$P(A/R)$

**Solved by**
**Labeeb Ahmad**

By Defination

$$P(A/R) = \frac{P(A \cap R)}{P(R)}$$

But $P(A \cap R)$ = Probability that urn A is selected and red ball drawn

So $P(A \cap R) = \frac{1}{3} \times \frac{5}{12} = \frac{5}{36}$

Hence

$$P(A|R) = \frac{5/36}{119/252}$$

$$= \frac{35}{119}$$

$$= 0.294$$

Solved by
Labeeb Ahmad

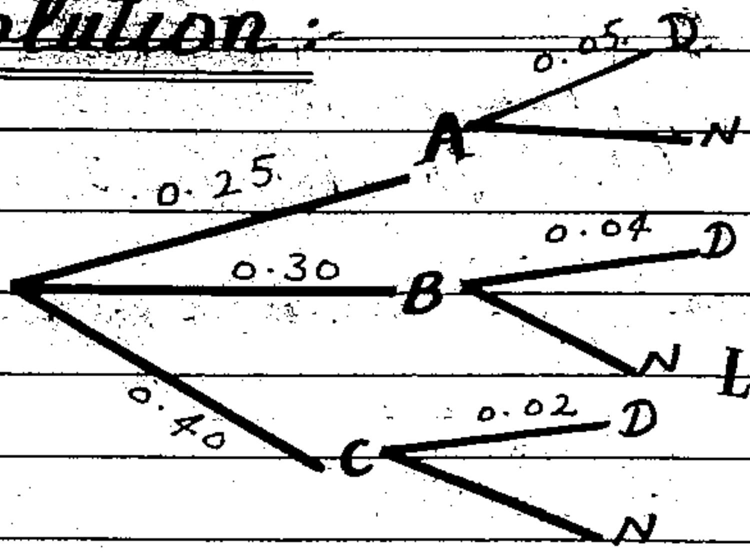
*——————————*

A/2011 , {A/2010 Q1(a)} pg=217

# Q.1(b)

In a bolt factory machines A, B and C manufacture 25, 30 and 40 percent of the total output respectively. Of their output 5, 4, 2 percent, respectively are defective bolts. A bolt is selected at random and is found to be defective. What is the probability that the bolt came from machine A?, B? and C?

## Solution:




Solved by Labeeb Ahmad

Probability that the item was produced by machine A.

Let x be the event that an item is defective.

By Bayes' theorem

$$P(A|x) = \frac{P(A)\,P(x|A)}{P(A)\,P(x|A) + P(B)\,P(x|B) + P(C)\,P(x|C)}$$

$$P(A/x) = \frac{(0.25)(0.05)}{(0.25)(0.05) + (0.30)(0.04) + (0.40)(0.02)}$$

$$= \frac{0.0125}{0.0325}$$

$$= \frac{125}{325}$$

**Solved by**
**Labeeb Ahmad**

$$P(A/x) = \frac{5}{13}$$

Now

The probability the item was produce by machine B.

$$P(B/x) = \frac{P(B)\,P(x/B)}{P(A)\cdot P(x/A) + P(B)\,P(x/B) + P(C)P(x/C)}$$

$$= \frac{(0.30)(0.04)}{(0.25)(0.05) + (0.30)(0.04) + (0.40)(0.02)}$$

$$= \frac{0.012}{0.0325} = \frac{24}{65}$$

Lastly

The Probability the item was produce by machine C.

$$P(C/x) = \frac{P(C)\,P(x/C)}{P(A)P(x/A) + P(B)\,P(x/B) + P(C)\cdot P(x/C)}$$

$$= \frac{(0.40)(0.02)}{(0.25)(0.05) + (0.30)(0.04) + (0.40)(0.02)}$$

$$= \frac{0.008}{0.0325} = \frac{16}{65}$$

A/2011



**Q2(a)** If X has probability density

$f(x) = k\exp(-3x)$ for $x > 0$.

(i) Find k and $P(\frac{1}{2} \leq X \leq 1)$

(ii) Find the distribution function of random variable X and use it to evaluate $P(\frac{1}{2} \leq X \leq 1)$

**Solution:-** Given $f(x) = Ke^{-3x}$

function will be p-d-f

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x) = 1$

Here $f(x) \geq 0$ if $K \geq 0$

And

$$\int_0^{\infty} Ke^{-3x} = 1 \qquad (\because x > 0)$$

$$K \int_0^{\infty} e^{-3x} = 1$$

$$K\left(\frac{e^{-3x}}{-3}\Big|_0^{\infty}\right) = 1$$

$$K\left(\frac{-1}{3e^{+3x}}\Big|_0^{\infty}\right) = 1$$

$$K\left(\frac{-1}{3e^{\infty}} + \frac{1}{3e^{0}}\right) = 1$$

Solved by Labeeb Ahmad

$K\left(0+\frac{1}{3}\right)=1$

$\boxed{K=3}$

Now $f(x)=3e^{-3x}$

$$P\left(\frac{1}{2}\leq x\leq 1\right)=\int_{1/2}^{1}3e^{-3x}\cdot dx$$

$$=3\left[\frac{e^{-3x}}{-3}\right]_{1/2}^{1}$$

$$=-\left[e^{-3}-e^{-\frac{3}{2}}\right]$$

$$=e^{-1.5}-e^{-3}$$

$$=0.2231-0.04978$$

$$=0.1733$$

**Solved by Labeeb Ahmad**

**(ii)**

To Compute the Distribution function we find $\left(F(x)=P(X<x)=\int_{-\infty}^{x}f(x)\cdot dx\right)$

For any $x<0$

$$F(x)=\int_{-\infty}^{0}0\cdot dx=0$$

For $x > 0$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} 3e^{-3x}\,dx$$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_{0}^{x}$$

$$= \frac{3}{-3}\left[e^{-3x} - e^{0}\right]$$

$$= -1\left[e^{-3x} - 1\right]$$

$$= 1 - e^{-3x}$$

Hence

$$F(X) = \begin{cases} 0 & ; \; x < 0 \\ 1 - e^{-3x} & ; \; x > 0 \end{cases}$$

To Find $P(\frac{1}{2} \leq x \leq 1)$ by using Distribution function.

$$P(\tfrac{1}{2} \leq x \leq 1) = F(1) - F(1/2)$$

$$= (1 - e^{-3}) - (1 - e^{-3/2})$$

$$= 1 - e^{-3} - 1 + e^{-3/2}$$

$$= e^{-3/2} - e^{-3}$$

$$= 0.2231 - 0.04978 = 0.1733$$

Solved by
Labeeb Ahmad

A/2011



**Q2(b)** Two Caplets are selected at random from a bottle containing three Aspirin two sedative and four laxative Caplets. If x and y are respectively the numbers of aspirin and sedative caplets included among the two Caplets drawn from the bottle, find the probabilities associative with all possible pair of value x and y. Find also all possible and also find marginal Probabilities.

**Solution:-**

See Paper A/2016 Q4 (b) at page "16"

A/2011

**Q3(a)** Prove that mean and varince of binomial distribution are $n\theta$ and $n\theta(1-\theta)$ respectively.

**Solution:-** See paper A/2012 Q2(a) at page "174"

A/2011 Pg = 223

**Q3(b)** A and B play a game in which A's probability of winning is 2/3. In a series of 8 games. What is the probability that A win

(i) Exactly 4 games
(ii) At least 4 games
(iii) 6 or more games
(iv) from 3 to 6 games

Solved by Labeeb Ahmad

**Solution:** Here for the Binomial Prob Dist

Here $n = 8$, $P = 2/3$

And $q = 1 - P = 1 - \frac{2}{3} = \frac{1}{3}$

Let $x$ denote the number of games won by A.

**(i)** $P(x=4) = \binom{8}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^4 = 0.1707$

**(ii)** $P(x \geq 4) = \sum_{x=4}^{8}\binom{8}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{8-x}$

$= \left[\binom{8}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^4\right] + \left[\binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3\right] + \left[\binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2\right]$

$+ \left[\binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^1\right] + \left[\binom{8}{8}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)^0\right]$

$= 0.1707 + 0.2731 + 0.2731 + 0.1560 + 0.0390$

$= 0.9118$

**(iii)** $P(x \geq 6) = \sum_{x=6}^{8} \binom{8}{x} \left(\frac{2}{3}\right)^{x} \left(\frac{1}{3}\right)^{8-x}$

$= \left[\binom{8}{6}\left(\frac{2}{3}\right)^{6}\left(\frac{1}{3}\right)^{2}\right] + \left[\binom{8}{7}\left(\frac{2}{3}\right)^{7}\left(\frac{1}{3}\right)^{1}\right] + \left[\binom{8}{8}\left(\frac{2}{3}\right)^{8}\left(\frac{1}{3}\right)^{0}\right]$

$= 0.2731 + 0.1560 + 0.0390$

$= 0.4681$

**(iv)** From 3 to 6

$P(3 \leq x \leq 6) = \sum_{x=3}^{6} \binom{8}{x} \left(\frac{2}{3}\right)^{x} \left(\frac{1}{3}\right)^{8-x}$

$= \left[\binom{8}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{5}\right] + \left[\binom{8}{4}\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{4}\right] + \left[\binom{8}{5}\left(\frac{2}{3}\right)^{5}\left(\frac{1}{3}\right)^{3}\right] + \left[\binom{8}{6}\left(\frac{2}{3}\right)^{6}\left(\frac{1}{3}\right)^{2}\right]$

$= 0.0682 + 0.1707 + 0.2731 + 0.2731$

$= 0.7852$

**Solved by**
**Labeeb Ahmad**

A /2011

Pg=225

# Q4(a) Write the properties of Normal Distribution

## Solution:

1— The function $f(x)$ defining the normal distribution is proper p.d.f. i.e $f(x) \geq 0$ and the total area under the normal curve is unity

2— The mean and variance of the normal distribution $\mu$ and $\sigma^2$ respectively

3— The median and the mode of the normal distribution are each equal to $\mu$ the mean of the normal distribution

4— The mean deviation of the normal distribution is approximately $\frac{4}{5}$ of its standard deviation

5— The normal curve has points of inflection which are equidistant from mean.

Solved by
Labeeb Ahmad

6- For the normal distribution, the odd order moments about the mean are all zero and the even order moments are given by

$$\mu_{2n} = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1\,\sigma^{2n}$$

7- If $X$ is $N(\mu, \sigma^2)$ and if $Y = a+bX$, then $Y$ is $N(a+b\mu, b^2\sigma^2)$

8- No matter what the value of $\mu$ and $\sigma$ are, areas under normal curve remain the certain fixed proportions within a specified number of S-D on either side of $\mu$.

For example

(i) $\mu \pm \sigma$ will always contain 68.26%.

(ii) $\mu \pm 2\sigma$ will always contain 95.44%.

(iii) $\mu \pm 3\sigma$ will always contain 99.73%.

Solved by
Labeeb Ahmad

Solved by
Labeeb Ahmad

A/2011



**Q4(b)** Find the probabilities that a random variable having the standard Normal Distribution will take on a value

(i) less than 1.72

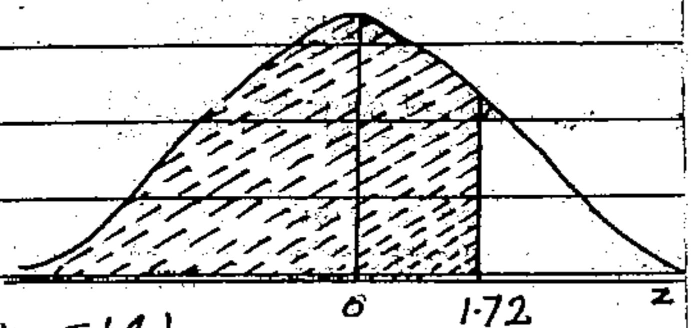
(ii) less than $-0.88$

(iii) between 1.30 and 1.75

(iv) between $-0.25$ and 0.45

## Solution:

**(i)** $P(Z < 1.72)$




$= 0.5 + P(0 \leq Z \leq 1.72)$

$= 0.5 + 0.4573$ (using Table)

$= 0.9573$

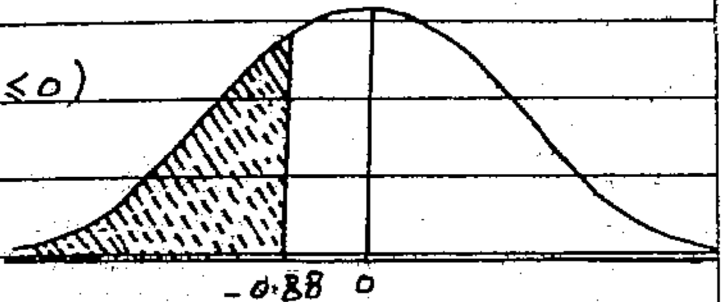
Solved by
Labeeb Ahmad

**(ii)** $P(Z < -0.88)$

$= 0.5 - P(-0.88 \leq Z \leq 0)$

$= 0.5 - 0.3106$

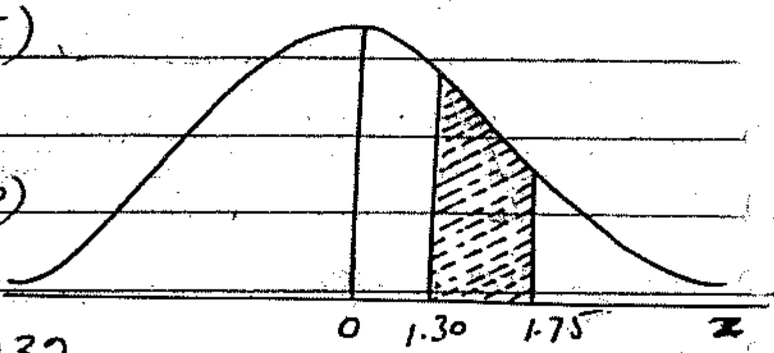
$= 0.1894$



**(iii)** between 1.30 and 1.75

$P(1.30 \leq Z \leq 1.75)$

$= P(0 \leq Z \leq 1.75) - P(0 \leq Z \leq 1.30)$




Using Table

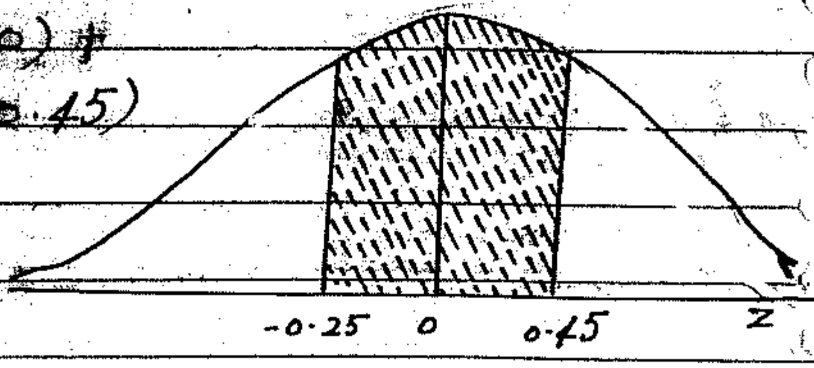
$= 0.4599 - 0.4032$

$= 0.0567$

**(iv)** between −0.25 and 0.45

$P(-0.25 < Z < 0.45)$

$= P(-0.25 \leq Z \leq 0) + P(0 \leq Z \leq 0.45)$




Using Table

$= 0.0987 + 0.1736$

$= 0.2723$

**Solved by**
**Labeeb Ahmad**

**Q5(a)** Write and Prove the properties of Mathematical Expectations.

**(1)** If $a$ is constant then $E(a)=a$

Proof: Let $X$ be a discreat r-v with p-d $[x_i, f(x_i)]$, $i=1,2,3 \cdots n$

Then $E(a)=\sum_{i=1}^{n} a f(x_i)$

$E(a)= a f(x_1)+a f(x_2)+a f(x_3)+ \text{——} + a f(x_n)$

$= a[f(x_1)+f(x_2)+f(x_3)+ \text{——} f(x_n)]$

$= a \sum f(x_i)$

Solved by Labeeb Ahmad

Since $\sum_{i=1} f(x_i) = 1$

so

$E(a) = a \cdot 1 = a$

Thus the Expected value of a Constant is Constant itself.

$E(ax+b)=aE(x)+b$ a & b are Constant

**(2)** If $x$ is a discreat r-v $X$ be $[x_i, f(x_i)]$ $i=1,2 — n$, we have

By Definition of expected value

$E(ax+b)=\sum_{i=1}^{n}(ax_i+b)f(x_i)$

$E(ax+b) = (ax_1+b)f(x_1) + (ax_2+b)f(x_2) + \cdots$
$\cdots (ax_n+b)f(x_n)$

$= ax_1 f(x_1) + b f(x_1) + ax_2 f(x_2) + b f(x_2)$
$+ \cdots ax_n f(x_n) + b f(x_n)$

$= a[x_1 f(x_1) + f(x_2) + \cdots x_n f(x_n)] +$
$b[f(x_1) + f(x_2) + f(x_3) + \cdots f(x_n)]$

$= a \sum_{i=1}^{n} x_i f(x_i) + b \sum_{i=1}^{n} f(x_i)$

$= a\, E(x) + b(1)$

$= aE(x) + b$

Solved by
Labeeb Ahmad

**(3)** The expected value of the sum of any two random variable is equal to the sum of their expected values i-e $E(X+Y) = E(X) + E(Y)$

**Proof**

Let x and y be two discreat r-v's defined on the same sample space S. Let x assume m values of $x_i$'s with probabilities $g(x_i)$'s and y assume values $y_i$'s with probabilities $h(y_i)$'s

The Sum $X+Y$ is a r-v taking the values $x_i + y_j$ and with Probabilities $f(x_i, y_j)$ for all Combinations of values of $i$ and $j$

Hence By Definition

$$E(X+Y) = \sum_i \sum_j (x_i + y_j) f(x_i, y_j)$$

$$= \sum_i \sum_j x_i f(x_i, y_j) + \sum_i \sum_j y_j f(x_i, y_j)$$

But $\sum_i \sum_j x_i f(x_i, y_j)$

$$= \sum_i x_i \sum_j f(x_i, y_j)$$

$$= \sum_i x_i \left[ f(x_i, y_1) + f(x_i, y_2) + \cdots f(x_i, y_n) \right]$$

$$= \sum_i x_i \, g(x_i)$$

$$= E(X)$$

Solved by
Labeeb Ahmad

Similarly

$$\sum_i \sum_j y_j \, f(x_i, y_j)$$

$$= y_j \sum f(x_i, y_j)$$

Solved by
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$$= \sum_j y_j \left[ f(x_1, y_j) + f(x_2, y_j) + \text{---} f(x_n, y_j) \right]$$

$$= \sum_j y_j \; h(y_j)$$

$$= E(Y)$$

Hence

$$E(X+Y) = E(X) + E(Y)$$

**(4)**, The expected value of the product of two independent r.v's is equal to the product of their expected values i.e

$$E(XY) = E(X).E(Y)$$

By Definition

$$E(XY) = \sum_i \sum_j x_i y_j f(x_i, y_j)$$

$$= \sum_i \sum_j x_i y_j \, g(x_i) . h(y_j)$$

$$= \sum_i x_i \, g(x_i) \cdot \sum_j y_j \, h(y_j)$$

$$= E(X) \, . \, E(Y)$$

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A|2011



**Q5(b)** Let x and Y be two discrete r-v with the following joint probability distribution.

| Y \ X | 2 | 4 |
|---|---|---|
| 1 | 0.10 | 0.15 |
| 3 | 0.20 | 0.30 |
| 5 | 0.10 | 0.15 |

Find $E(X)$, $E(Y)$, $E(2X-3Y)$ and $E(XY)$

**Solution:-** To determine the expected values of X and Y, we first find the marginal p-d $g(x)$ and $h(y)$ by adding over the columns and rows of two way table.

**Solved by Labeeb Ahmad**

| Y \ X | 2 | 4 | h(y) |
|---|---|---|---|
| 1 | 0.10 | 0.15 | 0.25 |
| 3 | 0.20 | 0.30 | 0.50 |
| 5 | 0.10 | 0.15 | 0.25 |
| g(x) | 0.40 | 0.60 | 1 |

Now $E(X) = \sum x_i \, g(x_i)$

$= 2(0.40) + 4(0.60)$

$= 0.80 + 2.40$

$= 3.2$

$E(Y) = \sum_i y_j\, h(y_j) = 1(0.25) + 3(0.50) + 5(0.25)$

**Written by**
**Labeeb Ahmad**
Mob:0341-7224008

$= 0.25 + 1.50 + 1.25$

$= 3$

$E(2X - 3Y) = 2E(X) - 3E(Y) = 2(3.2) - 3(3) = -2.6$

Since X and Y are independent therefor

$E(XY) = E(X) \cdot E(Y) = (3.2)(3) = 9.6$

A/2011

**Q6(a)** If the joint probability distribution of $x_1$ and $x_2$ is given by $f(x_1, x_2) = e^{-(x_1+x_2)}$ $x_1, x_2 \geq 0$ Find the probability density of $Y = \frac{x_1}{x_1+x_2}$

**Solution:** See Paper A/2015 Q7(a) at page "72"

A/2011

**Q6(b)** If x has Standard normal Distribution Find the density function of $z = x^2$

**Solution:-** See paper A/2016 Q7(b) at page "31"

A/2011

**Q7(a)** See paper A/2014 Q3(b) at page "96"

A/2011

**Q7(b)** See paper A/2013 Q2(a) at Pg = 137

A/2011

**Q8(a)** Central Limit Theorem

Solution See Paper A/2016 Q5(b) at Page "21"

A/2011

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**Q8(b)** Derive the mean and variance of Chi-square distribution.

**Solution:** The density function of ($\chi^2$) distribution

$$f(x) = \frac{x^{\frac{n}{2}-1} e^{-x/2}}{2^{n/2}\Gamma(n/2)} \qquad 0 < x < \infty$$

**Mean** $= E(X) = \int_0^\infty x \cdot \frac{x^{\frac{n}{2}-1} e^{-x/2}}{2^{n/2}\Gamma(n/2)} dx$

$$= \int_0^\infty \frac{x^{\frac{n}{2}} e^{-x/2}}{2^{n/2}\Gamma(n/2)} dx$$

put $\theta = x/2$

$d\theta = \frac{dx}{2}$ or $dx = 2d\theta$

$\theta \to 0$ when $x \to 0$

$\theta \to \infty$ when $x \to \infty$

$$E(x) = \mu = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty (2\theta)^{\frac{n}{2}} \cdot e^{-\theta} \cdot 2 \cdot d\theta$$

Solved by
Labeeb Ahmad

Pg = 236

$$E(X) = \frac{\cancel{2^{n/2}} \cdot 2}{\cancel{2^{n/2}}\,\Gamma(n/2)} \int_0^\infty \theta^{\frac{n}{2}+1-1} e^{-\theta}\, d\theta$$

$$E(X) = \frac{2}{\Gamma(n/2)} \Gamma\left(\frac{n}{2}+1\right)$$

using property $\Gamma(n) = (n-1)\Gamma(n-1)$

$$= \frac{2}{\cancel{\Gamma(n/2)}} \left(\frac{n}{2}\right) \cancel{\Gamma(n/2)}$$

$$= n\left(\frac{n}{2}\right)$$

| Mean = n |
|---|

Now $E(X^2) = \int_0^\infty \frac{x^2 \cdot x^{\frac{n}{2}-1} e^{-x/2}}{2^{n/2}\,\Gamma(n/2)}\, dx$

$$= \frac{1}{2^{n/2}\,\Gamma(n/2)} \int_0^\infty x^{\frac{n}{2}+1} e^{-x/2}\, dx$$

Solved by
Labeeb Ahmad

put $\frac{x}{2} = \theta$

$\frac{dx}{2} = d\theta$ or $dx = 2\, d\theta$

And $\theta \to \infty$ when $x \to \infty$

$\theta \to 0$ when $x \to 0$

So

$$E(X^2) = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty (2\theta)^{\frac{n}{2}+2-1} e^{-\theta} \cdot 2\,d\theta$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} (2^{\frac{n}{2}+1}) \cdot 2 \int_0^\infty \theta^{\frac{n}{2}-1} e^{-\theta}\, d\theta$$

$$= \frac{2^{n/2} \cdot 2^2}{2^{n/2}\Gamma(n/2)} \Gamma\left(\frac{n}{2}+2\right)$$

⇒ Using property of Γ function

$$\Gamma(n) = (n-1)(n-2)(n-3)\Gamma(n-3)$$

$$= \frac{4}{\Gamma(n/2)} \left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)\Gamma(n/2)$$

$$= 4\left[\frac{n+2}{2}\right]\left[\frac{n}{2}\right]$$

$$= n(n+2)$$

Solved by
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$$E(X^2) = n^2+2n$$

**Variance** $= E(x^2) - [E(x)]^2$

$$= n^2+2n-n^2$$

$$= 2n$$

A/2011 pg = 238

# Q9(a)

Suppose we want to determin on the basis of following data whether there is a relationship between the time, in minutes, it takes a secratary to Complete a Certain form in the morning and in the late afternoon.

| Morning (X) | 8.2 | 9.6 | 7.0 | 9.4 | 10.9 | 7.1 | 9.0 | 6.6 | 8.4 | 10.5 |
|---|---|---|---|---|---|---|---|---|---|---|
| Afternoon (Y) | 8.7 | 9.6 | 6.9 | 8.5 | 11.3 | 7.6 | 9.2 | 6.3 | 8.4 | 12.3 |

Compute and interpret the sample Correlation co-efficient.

**Solution:-** Correlation Co-efficient can be calculated.

$$r = \frac{\sum (X-\bar{X})(Y-\bar{Y})}{\sqrt{\sum (X-\bar{X})^2 \sum (Y-\bar{Y})^2}}$$

**Solved by Labeeb Ahmad**

OR

$$r = \frac{\sum XY - (\sum Y)/n}{\sqrt{\left[\sum X^2 - \frac{(\sum X)^2}{n}\right]\left[\sum Y^2 - \frac{(\sum Y)^2}{n}\right]}}$$

| X | Y | $X-\bar{X}$ | $(X-\bar{X})^2$ | $Y-\bar{Y}$ | $(Y-\bar{Y})^2$ | $(X-\bar{X})(Y-\bar{Y})$ |
|---|---|---|---|---|---|---|
| 8.2 | 8.7 | -0.47 | 0.2209 | -0.18 | 0.0324 | 0.0846 |
| 9.6 | 9.6 | 0.93 | 0.8649 | 0.72 | 0.5184 | 0.6696 |
| 7 | 6.9 | -1.67 | 0.7889 | -1.98 | 3.9204 | 3.3066 |
| 9.4 | 8.5 | 0.73 | 0.5329 | -0.38 | 0.1444 | -0.2774 |
| 10.9 | 11.3 | 2.23 | 4.9721 | 2.42 | 5.8564 | 5.3966 |
| 7.1 | 7.6 | -1.57 | 2.4649 | -1.28 | 1.6384 | 2.0096 |
| 9 | 9.2 | 0.33 | 0.1089 | 0.32 | 0.1024 | 0.1056 |
| 6.6 | 6.3 | -2.07 | 4.2849 | -2.58 | 6.6564 | 5.3406 |
| 8.4 | 8.4 | -0.27 | 0.0729 | -0.48 | 0.2304 | 0.1296 |
| 10.5 | 12.3 | 1.83 | 3.3489 | 3.42 | 11.6964 | 6.2586 |
| 86.7 | 88.8 | 0 | 19.661 | 0 | 30.796 | 23.024 |

$$\bar{X} = \frac{\sum X}{n} = \frac{86.7}{10} = 8.67$$

$$\bar{Y} = \frac{\sum Y}{n} = \frac{88.8}{10} = 8.88$$

**Solved by Labeeb Ahmad**

$$r = \frac{\sum (X-\bar{X})(Y-\bar{Y})}{\sqrt{\sum (X-\bar{X})^2 \sum (Y-\bar{Y})^2}}$$

$$r = \frac{23.024}{\sqrt{(19.661)(30.796)}} = \frac{23.024}{24.606}$$

$$r = 0.93$$

A|2011 Pg = 240

Q9(b) The following data show the number of bed rooms, the number of baths and the prices at which a random sample of eight one family houses sold in a certain large housing development.

| No. of Bed rooms $x_1$ | No. of baths $x_2$ | Price \$ $y$ |
|---|---|---|
| 3 | 2 | 78,800 |
| 2 | 1 | 74,300 |
| 4 | 3 | 83,800 |
| 2 | 1 | 74,200 |
| 3 | 2 | 79,700 |
| 2 | 2 | 74,900 |
| 5 | 3 | 88,400 |
| 4 | 2 | 82,900 |

Using the method of least squares to find a linear equation that will enable you to predict the average sales price of one family house in the given housing development in term of number of bed rooms and number of baths.

**Solution** The estimated multiple regression line is

$$\hat{Y} = \hat{\alpha} + \hat{\beta} X_1 + \hat{\gamma} X_2$$

By Method of least square Normal Equation

$$\left.\begin{aligned} \sum Y &= n\alpha + \beta \sum X_1 + \gamma \sum X_2 \\ \sum X_1 Y &= \alpha \sum X_1 + \beta \sum X_1^2 + \gamma \sum X_1 \sum X_2 \\ \sum X_2 Y &= \alpha \sum X_2 + \beta \sum X_1 \sum X_2 + \gamma \sum X_2^2 \end{aligned}\right\} \quad (1)$$

| $X_1$ | $X_2$ | $X_1^2$ | $X_2^2$ | $X_1X_2$ | $Y$ | $X_1Y$ | $X_2Y$ |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 9 | 4 | 6 | 78800 | 236400 | 157600 |
| 2 | 1 | 4 | 1 | 2 | 74300 | 148600 | 74300 |
| 4 | 3 | 16 | 9 | 12 | 83,800 | 335200 | 251400 |
| 2 | 1 | 4 | 1 | 2 | 74,200 | 148400 | 74200 |
| 3 | 2 | 9 | 4 | 6 | 79,700 | 239100 | 159400 |
| 2 | 2 | 4 | 4 | 4 | 74,900 | 149800 | 149800 |
| 5 | 3 | 25 | 9 | 15 | 88,400 | 442000 | 265200 |
| 4 | 2 | 16 | 4 | 8 | 82,900 | 331600 | 165800 |
| $\sum X_1 =$ | $\sum X_2 =$ | $\sum X_1^2 =$ | $\sum X_2^2 =$ | $\sum X_1X_2 =$ | $\sum Y =$ | $\sum X_1Y =$ | $\sum X_2Y =$ |
| 25 | 16 | 87 | 36 | 55 | 637000 | 2031100 | 1297700 |

**Solved by**
**Labeeb Ahmad**

We find $\alpha, \beta, \gamma$ by solving normal equations simultaneously

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} n & \Sigma X_1 & \Sigma X_2 \\ \Sigma X_1 & \Sigma X_1^2 & \Sigma X_1 X_2 \\ \Sigma X_2 & \Sigma X_1 X_2 & \Sigma X_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma Y \\ \Sigma X_1 Y \\ \Sigma X_2 Y \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 25 & 16 \\ 25 & 87 & 55 \\ 16 & 55 & 36 \end{bmatrix}^{-1} \begin{bmatrix} 63{,}7000 \\ 2031100 \\ 129{,}7700 \end{bmatrix}$$

det

$$A = \begin{bmatrix} 8 & 25 & 16 \\ 25 & 87 & 55 \\ 16 & 55 & 36 \end{bmatrix}$$

Solved by
Labeeb Ahmad

$$|A| = \begin{vmatrix} 8 & 25 & 16 \\ 25 & 87 & 55 \\ 16 & 55 & 36 \end{vmatrix}$$

$$|A| = 8(107) - 25(20) + 16(-17)$$

$$|A| = 84$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 87 & 55 \\ 55 & 36 \end{vmatrix} = 107$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 25 & 55 \\ 16 & 36 \end{vmatrix} = -20$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 25 & 87 \\ 16 & 55 \end{vmatrix} = -17$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 25 & 16 \\ 55 & 36 \end{vmatrix} = -20$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 8 & 16 \\ 16 & 36 \end{vmatrix} = 32$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 8 & 25 \\ 16 & 55 \end{vmatrix} = -40$$

Solved by
Labeeb Ahmad

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 25 & 16 \\ 87 & 55 \end{vmatrix} = -17$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 8 & 16 \\ 25 & 55 \end{vmatrix} = -40$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 8 & 25 \\ 25 & 87 \end{vmatrix} = 71$$

And

$$Adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Pg = 244

$$\text{Adj } A = \begin{bmatrix} 107 & -20 & -17 \\ -20 & 32 & -40 \\ -17 & -40 & 71 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{84} \begin{bmatrix} 107 & -20 & -17 \\ -20 & 32 & -40 \\ -17 & -40 & 71 \end{bmatrix}$$

Hence

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 107 & -20 & -17 \\ -20 & 32 & -40 \\ -17 & -40 & 71 \end{bmatrix} \begin{bmatrix} 63,7000 \\ 203,1100 \\ 129,7700 \end{bmatrix}$$

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{Y} \end{bmatrix} = \begin{bmatrix} 65191.7 \\ 4133.3 \\ 758.3 \end{bmatrix}$$

**Solved by Labeeb Ahmad**

Required least square equation is

$$\hat{Y} = 65191.7 + 4133.3X_1 + 758.3X_2$$

Each extra bedrooms adds on average 4133 \$ and each bath adds on average 758 \$ to the sale price of house

# PAPER A/2010

Solved by
Labeeb Ahmad

A/2010

## Q1(a)

In a bolt factory A, B and C manufacture 25, 35 and 40 of total output respectively. Of their outputs 5, 4 and 2 percent respectively are defective bolts. A bolt is selected at random and found to be defective. What is the probability that the bolt came from machine A or B or C ?

**Solution** :- See paper A/2011 Q1(b) at page "217"

A/2010

## Q1(b)

A|2010

Pg = 247

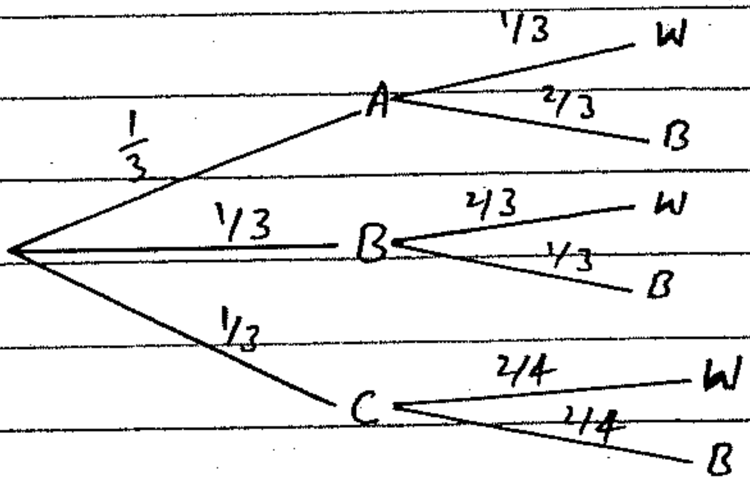
**Q2(a)** Three urns of same apperance have the following composition of

| Urn I | Urn II | Urn III |
|---|---|---|
| 1 white + 2 Black | 2W + 1B | 2W + 2B |

One of the urn is selected and a ball is drawn. It turns out to be white. What is the Probability that It came from III.

## Solution:-

| Urn I | Urn II | Urn III |
|---|---|---|
| 1W + 2B | 2W + 1B | 2W + 2B |




**Solved by Labeeb Ahmad**

$$P(W) = P(A) \cdot P(W/A) + P(B)\,P(W/B) + P(C)\,P(W/C)$$

Pg = 248

$$P(W) = \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{2}$$

$$P(W) = \frac{1}{9} + \frac{2}{9} + \frac{1}{6}$$

$$P(W) = 0.5$$

It Came from Urn III (C)

$$P(C/W) = \frac{P(C \cap W)}{P(W)}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$$

Solved by
Labeeb Ahmad

A/2010 , A/2008 Pg = 249

**Q2(b)** If x is binomial variate then prove that

$$\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

## Solution :-

$\mu_r = E(x-\mu)^r$ rth moment about mean

$$\mu_r = \sum_{x=0}^{n} (x-\mu)^r \binom{n}{x} p^x (1-p)^{n-x}$$

Differentiate w.r.t "p"

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} \binom{n}{x} \left[\left\{(x-np)^r\right\}\left\{\begin{matrix} p^x\{(n-x)(1-p)^{n-x-1}(-1)\} \\ + (1-p)^{n-x} x p^{x-1}\end{matrix}\right\}\right.$$

Ist as it Derivation of 2nd

$$+ \left\{p^x (1-p)^{n-x}\right\}\left\{r(x-np)^{r-1}(-n)\right\}$$

2nd as it Derivation of Ist

$$\frac{d\mu_r}{dp} = -\sum_{x=0}^{n} (x-np)^r \, {}^nC_x \, p^x (n-x)(q)^{n-x-1}$$

$$+\sum_{x=0}^{n} (x-np)^r \binom{n}{x} x p^{x-1} q^{n-x}$$

$$- rn \sum (x-np)^{r-1} \binom{n}{x} p^x q^{n-x}$$

Solved by
Labeeb Ahmad

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} {}^{n}C_x \left[(x-np)^r p^x q^{n-x}\right]\left[-(n-x)q^{-1} + xp^{-1}\right]$$

$$- rn\mu_{r-1}$$

$$= \sum_{x=0}^{n} {}^{n}C_x \left[(x-np)^r p^x q^{n-x}\right]\left[\frac{-n+x}{q} + \frac{x}{p}\right]$$

$$- rn\mu_{r-1}$$

$$= \sum_{x=0}^{n} {}^{n}C_x \left[(x-np)^r p^x q^{n-x}\right]\left[\frac{-pn+px+qx}{pq}\right]$$

$$- rn\mu_{r-1}$$

$$= \sum_{x=0}^{n} {}^{n}C_x \left[(x-np)^r p^x q^{n-x}\right]\left[\frac{-np+x}{pq}\right]$$

$$- rn\mu_{r-1}$$

$$= \sum_{x=0}^{n} {}^{n}C_x \left[(x-np)^{r+1} p^x q^{n-x}\right]\left[\frac{1}{pq}\right]$$

$$- rn\mu_{r-1}$$

$$\frac{d\mu_r}{dp} = \mu_{r+1}\left(\frac{1}{pq}\right) - rn\mu_{r-1}$$

$$\mu_{r+1}\left(\frac{1}{pq}\right) = rn\mu_{r-1} + \frac{d\mu_r}{dp}$$

$$\mu_{r+1} = pq\left[rn\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

Solved by
Labeeb Ahmad

Solved by
Labeeb Ahmad

A/2010 Pg = 251

Q3(a) see paper A/2012 Q4(b) at page "187"

Q3(b) Show that the normal curve has two point of inflexion which are equadistant from mean.

**Solution :-** the point of inflection by which we mean a point at which the concavity changes is obtained solving the equation $f''(x)=0$

Differentiating the function

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\cdot e^{\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$ , we get

$$f'(x)=\frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\right]\left(\frac{x-\mu}{\sigma}\right)\left(-\frac{2}{2}\right)\left(\frac{1}{\sigma}\right)$$

$$f'(x)=\frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}(-1)\left(\frac{x-\mu}{\sigma^2}\right)$$

$$f'(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\right)(x-\mu)$$

Now again Diff w.r.t "x"

Solved by
Labeeb Ahmad

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\right][1] + (x-\mu)\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}(-1)\left(\frac{x-\mu}{\sigma^2}\right)\right]$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left(1-\frac{(x-\mu)^2}{\sigma^2}\right)$$

Equating $f''(x)=0$, we are left with the following equation

$$1-\frac{(x-\mu)^2}{\sigma^2}=0$$

Solved by Labeeb Ahmad

which gives

$$(x-\mu)^2 = \sigma^2$$

or $\quad x-\mu = \pm\sigma$

$$x = \mu \pm \sigma$$

At these two point the value of function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\sigma}{\sigma}\right]^2}$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-1/2} = \frac{1}{\sigma\sqrt{2\pi e}}$$

Hence the two points of inflection of normal curve are $\left[\mu-\sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right]$ and $\left[\mu+\sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right]$. In other words, the point of inflection occurs on the right and the left of the mean at a distance equal to standard deviation.

A|2010 Pg = 253

Q4(a) See paper A|2013 Q7(a) at page "160"

Q4(b) Find the variance and Harmonic mean of $\beta_2$ Distribution

## Solution :-

The p.d.f of $\beta_2$

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 \le x \le \infty \quad m, n > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Recall $\beta_2(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \cdot dx \quad , m, n > 0$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

Solved by
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$$= \int_0^\infty x \frac{x^{m-1}}{B(m,n)} \frac{\cdot dx}{(1+x)^{m+n}}$$

$$= \frac{1}{B(m,n)} \int_0^\infty \frac{x^{(m+1)-1}}{(1+x)^{m+n}} \cdot dx$$

$$= \frac{1}{B(m,n)} \cdot B(m+1, n)$$

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$$E(x) = \frac{\dfrac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)}}{\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

$$= \frac{\Gamma(m+1)}{\Gamma(m+n+1)} \times \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$= \frac{m\,\Gamma(m)}{(m+n)\,\Gamma(m+n)} \times \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$E(x) = \frac{m}{m+n}$$

Now $$E(x^2) = \int_0^\infty x^2 \cdot \frac{1}{B(m,n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

Solved by
Labeeb Ahmad

$$= \frac{1}{B(m,n)} \int_0^\infty \frac{x^{(m+2)-1}}{(1+x)^{m+n}}\, dx$$

$$= \frac{B(m+2,n)}{B(m,n)}$$

$$= \frac{\dfrac{\Gamma(m+2)\,\Gamma(n)}{\Gamma(m+n+2)}}{\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

$$E(x^2) = \frac{\Gamma(m+2)}{\Gamma(m+n+2)} \times \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$= \frac{m(m+1)\,\cancel{\Gamma(m)}}{(m+n)(m+n+1)\,\cancel{\Gamma(m+n)}} \times \frac{\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m)}}$$

$$E(x^2) = \frac{m(m+1)}{(m+n)(m+n+1)}$$

**Variance** $= \sigma^2 = E(x^2) - [E(x)]^2$

$$\sigma^2 = \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m^2}{(m+n)^2}$$

$$= \frac{m(m+1)(m+n) - m^2(m+n+1)}{(m+n)^2(m+n+1)}$$

$$\sigma^2 = \frac{\cancel{m^3} + \cancel{m^2 n} + \cancel{m^2} + mn - \cancel{m^3} - \cancel{m^2 n} - \cancel{m^2}}{(m+n)^2(m+n+1)}$$

$$\sigma^2 = \frac{mn}{(m+n)^2(m+n+1)}$$

Solved by
Labeeb Ahmad

Now

**H-M** $= \left(\frac{1}{H}\right)^{-1}$ where $\frac{1}{H} = E\left(\frac{1}{x}\right)$

$$E\left[\frac{1}{x}\right] = \int_0^\infty \frac{1}{x} \cdot \frac{x^{m-1} \cdot dx}{B(m,n)\,(1+x)^{m+n}}$$

$$= \int_0^\infty \frac{x^{(m-1)-1}\, dx}{B(m,n)\,(1+x)^{m+n}}$$

$$E\left[\frac{1}{X}\right] = \frac{B(m-1,n)}{B(m,n)} \quad \text{①}$$

$\Rightarrow$ Take $B(m-1, n)$

$$B(m-1,n) = \frac{\Gamma(m-1)\cdot\Gamma(n)}{\Gamma(m+n-1)}$$

Multiply & Divide by $(m+n-1)(m-1)$

$$= \frac{(m+n-1)\left[(m-1)\Gamma(m-1)\right]\cdot\Gamma(n)}{(m-1)\left[(m+n-1)\Gamma(m+n-1)\right]}$$

$$= \frac{(m+n-1)}{m-1} \cdot \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{m+n-1}{m-1} \cdot B(m,n)$$

Solved by
Labeeb Ahmad

Put in ①

$$E(1/x) = \frac{\frac{m+n-1}{m-1}\cdot B(m,n)}{B(m,n)} = \frac{m+n-1}{m-1}$$

$$\textbf{H.M} = [E(1/x)]^{-1} = \frac{m-1}{m+n-1}$$

A|2010 Pg=257

## Q5(a)

If $\chi_1^2$ and $\chi_2^2$ are two independent Chi-square variates with $n_1$ and $n_2$ degree of freedom Then show that

$\dfrac{\chi_1^2}{\chi_2^2}$ is $\beta_2\left[\dfrac{n_1}{2}, \dfrac{n_2}{2}\right]$ variate

**Solution**:- The joint Probability of $\chi_1^2$ and $\chi_2^2$ is given by

$$f(\chi_1^2, \chi_2^2) = f(\chi_1^2) \cdot f(\chi_2^2)$$

$$f(\chi_1^2, \chi_2^2) = \left[\frac{e^{-\frac{\chi_1^2}{2}} (\chi_1^2)^{\frac{n_1}{2}-1}}{2^{n_1/2}\,\Gamma(n_1/2)}\right]\left[\frac{e^{-\frac{\chi_2^2}{2}} (\chi_2^2)^{\frac{n_2}{2}-1}}{2^{n_2/2}\,\Gamma(n_2/2)}\right]$$

$$= \frac{e^{-\frac{1}{2}(\chi_1^2+\chi_2^2)} (\chi_1^2)^{\frac{n_1}{2}-1} (\chi_2^2)^{\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}}\,\Gamma(n_1/2)\,\Gamma(n_2/2)}$$

$U = \dfrac{\chi_1^2}{\chi_2^2}$ and let $V = \chi_2^2$

$U = \dfrac{\chi_1^2}{V}$ or $\chi_1^2 = UV$

Solved by
Labeeb Ahmad

Now we transform $\chi_1^2, \chi_2^2$ into two new variable U and V. There for we use Jacobian transformation.

$$J = \frac{\partial(\chi_1^2, \chi_2^2)}{\partial(U,V)} = \begin{vmatrix} \frac{\partial \chi_1^2}{\partial U} & \frac{\partial \chi_1^2}{\partial V} \\ \frac{\partial \chi_2^2}{\partial U} & \frac{\partial \chi_2^2}{\partial V} \end{vmatrix}$$

$$J = \begin{vmatrix} V & U \\ 0 & 1 \end{vmatrix} = V$$

So

$\chi_1^2 = UV \quad \chi_2^2 = V$

$$F(U,V) = f(\chi_1^2, \chi_2^2) \; |J|$$

$$F(U,V) = \frac{e^{-\frac{1}{2}(UV+V)} (UV)^{\frac{n_1}{2}-1} V^{\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}} \Gamma(n_1/2) \Gamma(n_2/2)} (V)$$

$$= \frac{e^{-\frac{V}{2}(1+U)} U^{\frac{n_1}{2}-1} V^{\frac{n_1}{2}+\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}} \Gamma(n_1/2) \Gamma(n_2/2)}$$

So

$$F(U) = \int_0^\infty F(U,V) \, dV$$

Solved by
Labeeb Ahmad

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$$F(u) = \int_0^{\infty} \frac{e^{-\frac{v}{2}(1+u)} \, u^{\frac{n_1}{2}-1} \, v^{\frac{n_1}{2}+\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}} \, \Gamma(n_1/2) \cdot \Gamma(n_2/2)} \cdot dv$$

$$F(u) = \frac{u^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} \Gamma(n_1/2) \Gamma(n_2/2)} \int_0^{\infty} e^{-\frac{v}{2}(1+u)} \, v^{\frac{n_1}{2}+\frac{n_2}{2}-1} . dv \quad \text{---} (1)$$

Take $\int_0^{\infty} e^{-\frac{v}{2}(1+u)} \, v^{\frac{n_1}{2}+\frac{n_2}{2}-1} . dv$

let $\frac{v}{2}(1+u) = \theta$

or $v = \frac{2\theta}{1+u}$

Solved by
Labeeb Ahmad

$dv = \frac{2}{1+u} . d\theta$

$\theta \to 0$ when $v \to 0$
$\theta \to \infty$ when $v \to \infty$

Substituting these in above integral

$$= \int_0^{\infty} e^{-\theta} \left[\frac{2\theta}{1+u}\right]^{\frac{n_1}{2}+\frac{n_2}{2}-1} . \frac{2}{1+u} . d\theta$$

$$= \left[\frac{2}{1+u}\right]^{\frac{n_1}{2}+\frac{n_2}{2}} \int_0^{\infty} e^{-\theta} \, \theta^{\frac{n_1}{2}+\frac{n_2}{2}-1} . d\theta$$

$$=\left(\frac{2}{1+U}\right)^{\frac{n_1}{2}+\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)$$

Putting this in (1)

$$F(U) = \frac{U^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} \Gamma(n_1/2)\, \Gamma(n_2/2)\, (1+U)^{\frac{n_1}{2}+\frac{n_2}{2}}} (2)^{\frac{n_1}{2}+\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)$$

$$= \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma(n_2/2)} \frac{U^{\frac{n_1}{2}-1}}{(1+U)^{\frac{n_1}{2}+\frac{n_2}{2}}}$$

$$F(U) = \frac{U^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)(1+U)^{\frac{n_1}{2}+\frac{n_2}{2}}}$$

It is a density function of $\beta_2$-distribution

Hence it shows that $U = \frac{\chi_1^2}{\chi_2^2}$ is $\beta_2\left(\frac{n_1}{2},\frac{n_2}{2}\right)$ variate.

**Solved by**
**Labeeb Ahmad**

A/2010, S/2010, A/2009 Pg = 261

## Q5(b) Find the mode and skewness of $\chi^2$ distribution

**Solution :-** the p.d.f of $\chi^2$ distribution is

$$f(x) = \frac{1}{2^{n/2}\,\Gamma(n/2)} \cdot e^{-\frac{x}{2}} \cdot x^{\frac{n}{2}-1}$$

Diff w.r.t "x"

$$f'(x) = \frac{1}{2^{n/2}\,\Gamma(n/2)} \left[ e^{-\frac{x}{2}} \cdot \left\{ \left(\frac{n}{2}-1\right) x^{\frac{n}{2}-2} \right\} + \left\{ x^{\frac{n}{2}-1} \right\} \left\{ e^{-\frac{x}{2}} \left(-\frac{1}{2}\right) \right\} \right]$$

$$f'(x) = \frac{1}{2^{n/2}\,\Gamma(n/2)} e^{-\frac{x}{2}} x^{\frac{n}{2}-1} \left[ \left(\frac{n}{2}-1\right) \cdot x^{-1} - \frac{1}{2} \right]$$

$$f'(x) = f(x) \left[ \left(\frac{n}{2}-1\right) \cdot x^{-1} - \frac{1}{2} \right] \quad \text{--- (i)}$$

Equating $f'(x) = 0$

$$f(x) \left[ \left(\frac{n}{2}-1\right) x^{-1} - \frac{1}{2} \right] = 0$$

$$f(x) \neq 0$$

So $\left(\frac{n}{2}-1\right) x^{-1} - \frac{1}{2} = 0$

$$\left[\frac{n-2}{2}\right] x^{-1} = \frac{1}{2}$$

**Solved by. Labeeb Ahmad**

$$(n-2)x^{-1} = 1$$

$$\boxed{x = n-2} \qquad \because n > 2$$

Now again differentiate equation (i) w.r.t "x"

$$f''(x) = f(x)\frac{d}{dx}\left\{\left(\frac{n}{2}-1\right)x^{-1}-\frac{1}{2}\right\} + \left\{\left(\frac{n}{2}-1\right)x^{-1}-\frac{1}{2}\right\}\frac{d}{dx}(f(x))$$

$$= f(x)\left[\left(\frac{n}{2}-1\right)(-x^{-2})\right] + \left[\left(\frac{n}{2}-1\right)x^{-1}-\frac{1}{2}\right]f'(x)$$

$$= f(x)\left[\left(\frac{n}{2}-1\right)(-x^{-2})\right] + \left[\left(\frac{n}{2}-1\right)x^{-1}-\frac{1}{2}\right]f(x)\left[\left(\frac{n}{2}-1\right)x^{-1}-\frac{1}{2}\right]$$

$$f''(x) = f(x)\left[\left(\frac{n}{2}-1\right)(-x^{-2}) + \left\{\left(\frac{n}{2}-1\right)x^{-1}-\frac{1}{2}\right\}^2\right]$$

at $x = n-2$

$$= f(x)\left[\left(\frac{n-2}{2}\right)\left(\frac{-1}{(n-2)^2}\right) + \left\{\left(\frac{n-2}{2}\right)\left(\frac{1}{n-2}\right)-\frac{1}{2}\right\}^2\right]$$

$$= f(x)\left[\frac{-1}{2(n-2)} + \left(\frac{1}{2}-\frac{1}{2}\right)^2\right]$$

$$= f(x)\left[\frac{-1}{2(n-2)} + 0\right]$$

**Solved by**
**Labeeb Ahmad**

$$f''(x) = -\frac{f(x)}{2(n-2)} < 0 \qquad (\because f(x) \neq 0)$$

also $(\because n > 2)$

Hence mode of $\chi^2$ distribution is $n-2$.

**Skewness** See A/2013 Q5(b) at pg "154"

5/2010



**Q** If $\chi_1^2$ and $\chi_2^2$ are two independent $\chi^2$-variate with $n_1$ and $n_2$ degree of freedom. Then

$$\frac{\chi_1^2}{\chi_1^2+\chi_2^2} \text{ is } \beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \text{ variate.}$$

**Solution:** The joint Probability of $\chi_1^2$ and $\chi_2^2$ is given by

$$f(\chi_1^2, \chi_2^2) = f(\chi_1^2)\cdot f(\chi_2^2)$$

$$= \left[\frac{e^{-\frac{\chi_1^2}{2}} (\chi_1^2)^{\frac{n_1}{2}-1}}{2^{n_1/2}\Gamma(n_1/2)}\right]\left[\frac{e^{-\frac{\chi_2^2}{2}} (\chi_2^2)^{\frac{n_2}{2}-1}}{2^{n_2/2}\Gamma(n_2/2)}\right]$$

$$= \frac{e^{-\frac{1}{2}[\chi_1^2+\chi_2^2]} (\chi_1^2)^{\frac{n_1}{2}-1} \cdot (\chi_2^2)^{\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma(n_1/2)\,\Gamma(n_2/2)}$$

$$U = \frac{\chi_1^2}{\chi_1^2+\chi_2^2} \quad \text{and let } V = \chi_1^2+\chi_2^2$$

$$U = \frac{\chi_1^2}{V} \quad \text{or} \quad \chi_1^2 = UV$$

Solved by Labeeb Ahmad

And $V = \chi_1^2+\chi_2^2 = UV+\chi_2^2$

$$V - UV = \chi_2^2$$

$$\text{or} \quad \chi_2^2 = V(1-U)$$

$$J = \frac{\partial(\chi_1^2, \chi_2^2)}{(U,V)} = \begin{vmatrix} \frac{\partial \chi_1^2}{\partial U} & \frac{\partial \chi_1^2}{\partial V} \\ \frac{\partial \chi_2^2}{\partial U} & \frac{\partial \chi_2^2}{\partial V} \end{vmatrix}$$

$$= \begin{vmatrix} V & U \\ -V & (1-U) \end{vmatrix}$$

$$= V(1-U) - (U)(-V)$$

$$= V - UV + UV$$

$$J = V$$

Now we transform $\chi_1^2, \chi_2^2$ into two new variable $U$ & $V$ there for we apply Jacobian transformation by putting $\chi_1^2 = UV$ and $\chi_2^2 = V(1-U)$

$$F(U,V) = f(\chi_1^2 \cdot \chi_2^2) \cdot |J|$$

$$= \frac{e^{-\frac{1}{2}[V]} (UV)^{\frac{n_1}{2}-1} \{V(1-U)\}^{\frac{n_2}{2}-1} (V)}{2^{\frac{n_1+n_2}{2}} \Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2})}$$

$$F(U,V) = \frac{e^{-\frac{V}{2}} U^{\frac{n_1}{2}-1} (1-U)^{\frac{n_2}{2}-1} V^{\frac{n_1}{2}+\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}} \Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2})}$$

$$F(U) = \int_0^\infty F(U,V) \cdot dV$$

Solved by
Labeeb Ahmad

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$$F(U) = \frac{U^{\frac{n_1}{2}-1}(1-U)^{\frac{n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} \underbrace{\int_0^\infty e^{-\frac{V}{2}} V^{\frac{n_1}{2}+\frac{n_2}{2}-1} dV}_{I_1}$$

Take $I_1$ and

put $\frac{V}{2} = \theta$

$V = 2\theta$

$dV = 2\,d\theta$

$$\begin{cases} \theta \to 0 & \text{when } V \to 0 \\ \theta \to \infty & \text{when } V \to \infty \end{cases}$$

$$I_1 = \int_0^\infty e^{-\theta} (2\theta)^{\frac{n_1}{2}+\frac{n_2}{2}-1} \cdot 2 \cdot d\theta$$

$$= \int_0^\infty e^{-\theta}\, 2^{\frac{n_1}{2}+\frac{n_2}{2}-1} (\theta)^{\frac{n_1}{2}+\frac{n_2}{2}-1}\, 2 \cdot d\theta$$

$$= 2^{\frac{n_1}{2}+\frac{n_2}{2}} \int_0^\infty e^{-\theta}\, \theta^{\frac{n_1}{2}+\frac{n_2}{2}-1}\, d\theta$$

$$= 2^{\frac{n_1}{2}+\frac{n_2}{2}}\, \Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)$$

Solved by Labeeb Ahmad

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Putting this in F(U)

$$F(U) = \frac{U^{\frac{n_1}{2}-1} (1-U)^{\frac{n_2}{2}-1} \cancel{2^{\frac{n_1}{2}+\frac{n_2}{2}}} \Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\cancel{2^{\frac{n_1+n_2}{2}}} \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)}$$

$$F(U) = \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \cdot U^{\frac{n_1}{2}-1} (1-U)^{\frac{n_2}{2}-1}$$

$$F(U) = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} U^{\frac{n_1}{2}-1} (1-U)^{\frac{n_2}{2}-1}$$

It is a density function of Beta (Ist kind) distribution

Hence this shows that $U = \frac{\chi_1^2}{\chi_1^2 + \chi_2^2}$ is $\beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ variate.

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**Q** Find the mode of F-distribution

## Solution:

As

$$f(F) = \frac{(n/m)^{\frac{n}{2}}}{\beta\left(\frac{n}{2}, \frac{m}{2}\right)} \frac{F^{\frac{n}{2}-1}}{\left(1+\frac{n}{m}F\right)^{\frac{n+m}{2}}}$$

$$f(F) = \frac{A\, F^{\frac{n}{2}-1}}{\left(1+\frac{n}{m}F\right)^{\frac{n+m}{2}}} \quad \text{where } A = \frac{\left(\frac{n}{m}\right)^{\frac{n}{2}}}{\beta\left(\frac{n}{2}, \frac{m}{2}\right)}$$

Taking Log both side

$$\log f(F) = \log A + \left(\frac{n}{m} - 1\right)\log F - \left(\frac{n+m}{2}\right)\log\left(1+\frac{nF}{m}\right)$$

Differentiate w-r-t F

$$\frac{1}{f(F)} f'(F) = 0 + \left(\frac{n}{2} - 1\right)\frac{1}{F} - \frac{\left(\frac{n+m}{2}\right)}{\left(1+\frac{nF}{m}\right)} \cdot \left(\frac{n}{m}\right)$$

$$f'(F) = f(F)\left[\frac{n-2}{2F} - \frac{(n+m)n}{2(m+nF)}\right] \quad (1)$$

Equating $f'(F) = 0$

Solved by
Laheeb Ahmad

Pg = 268

$$\frac{n-2}{2F} - \frac{(n+m)n}{2(m+nF)} = 0$$

$$\frac{n-2}{2F} = \frac{(n+m)n}{2(m+nF)}$$

$$F = \frac{(m+nF)(n-2)}{(n+m)n}$$

$$(n+m)nF = n^2F + nm - 2nF - 2m$$

$$n^2F + mnF = n^2F + mn - 2nF - 2m$$

$$mnF + 2nF = mn - 2m$$

$$nF[m+2] = m(n-2)$$

$$F = \frac{m(n-2)}{n(m+2)}$$

Again equ (1) Diff w.r.t F

$$f''(F) = f'(F)\left[\frac{n-2}{2F} - \frac{(n+m)n}{2(m+nF)}\right] + f(F)\left[\frac{(n-2)}{-2F^2} + \frac{(n+m)n}{2}(m+nF)^{-2}n\right]$$

Solved by
Labeeb Ahmad

pg = 289

$$f''(F) = f'(F)\left[\frac{n-2}{2F} - \frac{(n+m)n}{2(m+nF)}\right] +$$

**Solved by Labeeb Ahmad**

$$f(F)\left[\frac{n-2}{-2F^2} + \frac{n^2(n+m)}{2(m+nF)^2}\right]$$

at $F = \dfrac{m(n-2)}{n(m+2)}$

since $f'(F) = 0$

So

$$f''(F) = 0 + f(F)\left[\frac{n-2}{-2\left[\frac{m(n-2)}{n(m+2)}\right]^2} + \frac{n^2(n+m)}{2\left[m+n\left\{\frac{m(n-2)}{n(m+2)}\right\}\right]^2}\right]$$

$$= f(F)\left[\frac{(n-2)\cdot n^2(m+2)^2}{-2m^2(n-2)^2} + \frac{n^2(n+m)}{2m^2[m+2+n-2]^2}\right]$$

$$= f(F)\left[\frac{n^2(m+2)^2}{-2m^2(n-2)} + \frac{n^2(m+n)}{2m^2(m+n)^2}\right]$$

$$= f(F)\left[\frac{n^2(m+2)^2}{-2m^2(n-2)} + \frac{n^2}{2m^2(m+n)}\right]$$

$$= f(F)\left[-\frac{n^2}{2m^2}\left(\frac{(m+2)^2}{(n-2)} - \frac{1}{m+n}\right)\right]$$

Pg = 270

$$f''(F) = -\frac{n^2}{2m^2} f(F)\left[\frac{m^2+4+2m}{(n-2)} - \frac{1}{m+n}\right]$$

$$= \frac{-n^2}{2m^2} f(F)\left[\frac{(m^2+4+2m)(m+n) - (n-2)}{(n-2)(m+2)}\right]$$

$$= \frac{-n^2}{2m^2} f(F)\left[\frac{m^3+m^2n+4m+4n+2m^2+2mn-n+2}{(n-2)(m+2)}\right]$$

For $m > 0$ and $n > 2$

$f''(F) < 0$

Hence Mode of F-distribution is $\frac{m(n-2)}{n(m+2)}$

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5/2010



# Q

Show that the Co-efficient of Correlation "r" between two variables X and Y is given by

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\,\sigma_x\,\sigma_y}$$

where $\sigma_x^2$, $\sigma_y^2$ and $\sigma_{x-y}^2$ are variances of X, Y and X-Y respectively

## Solution :-

$$\text{As } \sigma_{x-y}^2 = Var[X-Y]$$

$$= E[(X-Y) - (\overline{X-Y})]^2$$

$$= E[(X-Y) - (\bar{X}-\bar{Y})]^2$$

$$= E[(X-\bar{X}) - (Y-\bar{Y})]^2$$

**Solved by**
**Labeeb Ahmad**

$$= E\left[(X-\bar{X})^2 + (Y-\bar{Y})^2 - 2(X-\bar{X})(Y-\bar{Y})\right]$$

$$= E(X-\bar{X})^2 + E(Y-\bar{Y})^2 - 2E(X-\bar{X})(Y-\bar{Y})$$

$$\sigma_{x-y}^2 = \sigma_x^2 + \sigma_y^2 - 2\,Cov(x,y) \quad \text{--- (i)}$$

As $\gamma = \dfrac{Cov(x,y)}{\sigma_x\,\sigma_y}$

And $\gamma\,\sigma_x\sigma_y = Cov(x,y)$

put in (i)

$$\sigma_{x-y}^2 = \sigma_x^2 + \sigma_y^2 - 2\gamma\,\sigma_x\sigma_y$$

$$2\gamma\,\sigma_x\sigma_y = \sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2$$

$$\gamma = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\,\sigma_x\sigma_y}$$

Hence prove

**Solved by**
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Labeeb

5/2010

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**Q** Show that for t- distribution with "n" degree of freedom **M-D** about mean is given by

$$\frac{\sqrt{n}\ \Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\ \Gamma(n/2)}$$

**Solution**:- By Definition

$M\text{-}D = E\left[|t - E(t)|\right]$

$E[|t|]$ Because $E(t)=0$.

And the density function of "t" -Dist

$$f(t) = \frac{1}{\sqrt{n}\ \beta\left(\frac{n}{2},\frac{1}{2}\right)}\left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)}$$

So $M\text{-}D = E(|t|) = \int_{-\infty}^{\infty} |t| \cdot f(t) \cdot dt$

$$= \int_{-\infty}^{0} -t\, f(t) \cdot dt + \int_{0}^{\infty} t\ f(t) \cdot dt$$

$$= \int_{0}^{\infty} t\, f(t)\, dt + \int_{0}^{\infty} t \cdot f(t) \cdot dt$$

Solved by
Labeeb Ahmad

$$M.D = 2\int_0^\infty t \cdot f(t) \cdot dt$$

$$= 2\int_0^\infty t \frac{1}{\sqrt{n}\ \beta\left(\frac{n}{2}, \frac{1}{2}\right)} \left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)} \cdot dt$$

$$= \frac{2}{\sqrt{n}\ \beta\left(\frac{n}{2}, \frac{1}{2}\right)} \int_0^\infty t \cdot \left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)} \cdot dt$$

↓ $I_1$

Take $I_1$ and
put $\frac{t^2}{n} = \theta$

$$t = \sqrt{n\theta}$$

$$dt = \sqrt{n} \cdot \frac{1}{2\sqrt{\theta}} \cdot d\theta$$

$\theta \to 0$ when $t \to 0$

$\theta \to \infty$ when $t \to \infty$

$$I_1 = \int_0^\infty \sqrt{n\theta}\ (1+\theta)^{-\frac{1}{2}(n+1)} \frac{\sqrt{n}}{2\sqrt{\theta}} d\theta$$

$$= \frac{n}{2}\int_0^\infty \frac{\theta^0}{(1+\theta)^{\frac{n}{2}+\frac{1}{2}}} \cdot d\theta$$

$$= \frac{n}{2}\int_0^\infty \frac{\theta^{1-1}}{(1+\theta)^{\frac{n}{2}-\frac{1}{2}+1}} \cdot d\theta$$

Solved by
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$$I_1 = \frac{n}{2} \beta_2\left(\frac{n}{2} - \frac{1}{2}, 1\right)$$

$$I_1 = \frac{n}{2} \beta_2\left(\frac{n-1}{2}, 1\right)$$

Putting the value $I_1$

$$M\text{-}D = \frac{2}{\sqrt{n}\, \beta\left(\frac{n}{2}, \frac{1}{2}\right)} \cdot \frac{n}{2} \beta\left(\frac{n-1}{2}, 1\right)$$

$$= \frac{\sqrt{n}\, \dfrac{\Gamma\left(\frac{n-1}{2}\right) \Gamma(1)}{\Gamma\left(\frac{n-1}{2}+1\right)}}{\dfrac{\Gamma\left(\frac{n}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}}$$

$$M\text{-}D = \frac{\sqrt{n}\, \Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\, \Gamma(n/2)}$$ Hence prove

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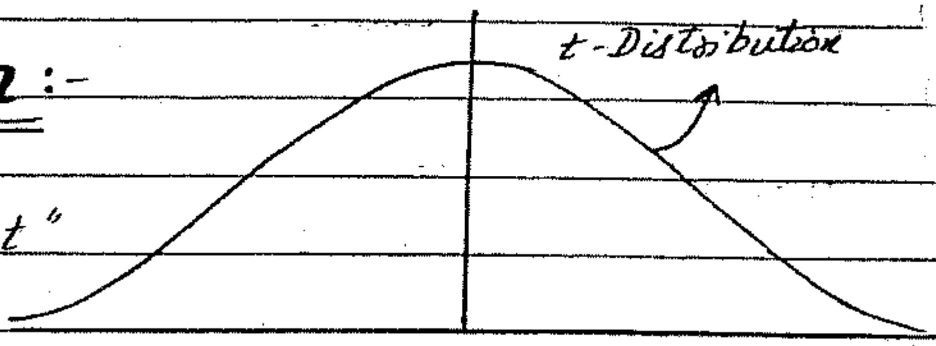
from sharaq pur

5/2010 Pg = 276

**Q** Prove all odd order moments of "t" distribution is zero.

## Solution :-




A r-v "t" has a symmetric Distribution. It means that "t" and "-t" have the same distribution.

This mean that for any function "f". If $E[f(t)]$ exist then so does $E[f(-t)]$ and they have equal

So

$$E[t^n] = E[(-t)^n]$$

$$E(t^n) = E[(-1)^n t^n]$$

$$E(t^n) = -E(t^n) \quad \because n \text{ is odd}$$

$$E(t^n) + E(t^n) = 0$$

$$2E(t^n) = 0$$

Solved by
Labeeb Ahmad

So $E(t^n) = 0$

where $n$ is Odd.

# PAPER A/2008

**Solved by**
**Labeeb Ahmad**

A/2008

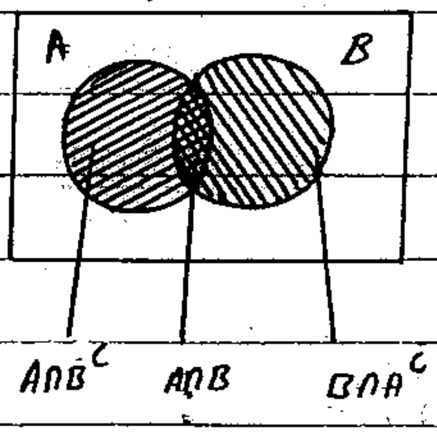
## Q1 (a) State and Prove addition law of Probability

Statment:

⇒ If A and B are any two events define in a sample space S, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof :-

The event $A \cup B$ may be written as the union of two mutually exclusive events A and $B \cap A^c$. (See diagram) that is




$A \cup B$ is shaded

$A \cup B = A \cup (B \cap A^c)$

Then $P(A \cup B) = P(A) + P(B \cap A^c)$

Again the event B may also be decomposed into two mutually exclusive events as

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$$B = (A \cap B) \cup (\bar{A} \cap B)$$

Thus $P(B) = P(A \cap B) + P(\bar{A} \cap B)$

Subtracting this result from the former, we get

$$P(A \cup B) - P(B) = P(A) - P(A \cap B)$$

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

---

**Q1(b)** If A and B are two independent events then prove that:

(i) $P(A \cap B) = P(A) . P(B)$

(ii) A and $B^c$ are independent.

(iii) $A^c$ and $B^c$ are independent.

**Proof: (i)** Since A and B are independent events, therefor

$P(A) = P(A/B)$ and $P(B) = P(B/A)$

Substituting these result in the genral rule of multiplication we obtain

$$P(A \cap B) = P(A)P(B)$$

**(ii)**

The events $A \cap B$ and $A \cap B^c$ are mutully exclusive and their union is A.

i.e $A = (A \cap B) \cup (A \cap B^c)$

There for $P(A) = P(A \cap B) + P(A \cap \bar{B})$

or $P(A \cap B^c) = P(A) - P(A \cap B)$

$= P(A) - P(A) \cdot P(B)$ $\because$ A & B are independent

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**Labeeb Ahmad**

$= P(A)\ (1 - P(B))$

$= P(A)\ P(B^c)$

Hence A and $B^c$ are independent.

**(iii)**

Using De morgan's Law

$A^c \cap B^c = (A \cup B)^c$

So $P(A^c \cap B^c) = P(A \cup B)^c$

$= 1 - P(A \cup B)$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A) \cdot P(B)$

$= [1 - P(A)] - P(B)[1 - P(A)]$

$= [1 - P(A)][1 - P(B)]$

$= P(A^c) \cdot P(B^c)$

This shows that $A^c$ and $B^c$ are independent.

**Q2(a)** Derive the normal Distribution as a limiting case of binomal distribution.

**Solution:-** See Paper A/2014 Q3(b) at page "96"

**Q2(b)** Show that Mean = Median = mode for normal Distribution.

Solved by
Labeeb Ahmad

**Solution:** See Paper A/2014 Q6(a) at Page "108"

A/2008, A/2004

**Q3(a)** Find the mean and variance of negative Binomial Distribution.

**Solution:-** we find mean and variance by driving m.g.f of negative binomial Distribution.

AS $P(X=x) = \binom{x-1}{k-1} p^k q^{x-k}$

Replacing $x-k$ by $x'$

$P(X=x') = \binom{x'+k-1}{k-1} p^k q^{x'}$

OR

$P(X=x) = \binom{x+k-1}{k-1} p^k q^x$

The m.g.f about origin is

$M_0(t) = E(e^{tx})$

**Solved by Labeeb Ahmad**

$= \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} p^k q^x \cdot e^{tx}$

$= p^k \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} (qe^t)^x$

$= p^k (1-qe^t)^{-k}$

Now Mean $= E(X) = \frac{d}{dt} M_0(t) \Big|_{t \to 0}$

$= \lim_{t \to 0} p^k \left[ -k(1-qe^t)^{-k-1}(-qe^t) \right]$

$= \lim_{t \to 0} p^k \left[ kqe^t (1-qe^t)^{-k-1} \right]$

$= p^k \left[ kqe^0 (1-qe^0)^{-k-1} \right]$

$= p^k k \cdot q (1-q)^{-k-1}$

$$= p^k \cdot kq\, p^{-k-1} \qquad \because 1-q = p$$

$$= kq\, p^{-1}$$

$$\boxed{\text{Mean} = \frac{kq}{p}}$$

**Solved by**
**Labeeb Ahmad**

$$\text{Variance} = \sigma^2 = E(x^2) - [E(x)]^2$$

where $E(x^2) = \lim_{t\to 0} \frac{d^2}{dt^2} M_0(t)$

Solved by Labeeb Ahmad

$$= p^k \frac{d}{dt}\left[kq\, e^t (1-qe^t)^{-k-1}\right]_{t\to 0}$$

$$= p^k \cdot kq\left[e^t \cdot (-k-1)(1-qe^t)^{-k-2}(-qe^t) + (1-qe^t)^{-k-1} \cdot e^t\right]_{t\to 0}$$

$$= p^k kq\left[e^0 (-k-1)(-qe^0)(1-qe^0)^{-k-2} + ((1-(q)e^0)^{-k-1} \cdot e^0\right]$$

$$= p^k kq\left[q(1+k)(1-q)^{-k-2} + (1-q)^{-k-1}\right]$$

$$= p^k kq\left[q(1+k)\, p^{-k-2} + p^{-k-1}\right]$$

$$= kq^2(1+k)\, p^{-2} + kq\, p^{-1}$$

$$= \frac{kq^2(1+k)}{p^2} + \frac{kq}{p}$$

Thus $\sigma^2 = \frac{kq}{p} + \frac{q^2k(1+k)}{p^2} - \left(\frac{kq}{p}\right)^2$

$$\frac{kpq + k^2q^2 + kq^2 - k^2q^2}{p^2} = \frac{kq(p+q)}{p^2} = \frac{kq}{p^2}$$

A|2008 Pg = 283

**Q3(b)** If X is Possion Distribution then show that

$$\mu_{r+1} = m\left(r\mu_{r-1} + \frac{r}{rm}(\mu_r)\right)$$

**Solution:-** See Paper A|2014 Q4(b) at page "101"

A|2008

**Q4(a)** Find the Mean and standard deviation of exponential Distribution.

**Solution** See Paper A|2012 Q6(a) at Page "194"

A|2008

**Q4(b)** If Mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively, then find $P(X \geq 1)$

Solved by
Labeeb Ahmad

**Solution:-** As we know

mean of Binomial Distribution $= np = 4$

variance of Binomial Distribution $= npq = \frac{4}{3}$

$np = 4$ , $npq = \frac{4}{3}$

$\cancel{4}q = \frac{\cancel{4}}{3}$

$\boxed{q = \frac{1}{3}}$

$p = 1 - q$

$p = 1 - \frac{1}{3} = \frac{3-1}{3} = \frac{2}{3}$

$\boxed{p = \frac{2}{3}}$

**Solved by Labeeb Ahmad**

As $np = 4$

$n \cdot \frac{2}{3} = 4$

$n = 4 \times \frac{3}{2} \Rightarrow \boxed{n = 6}$

$$P(x \geq 1) = \sum_{x=1}^{6} {}^8C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}$$

$$= \left[{}^8C_1 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5\right] + \left[{}^8C_2 \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4\right]$$

$$+ \left[{}^8C_3 \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3\right] + \left[{}^8C_4 \left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2\right] + \left[{}^8C_5 \left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1\right]$$

$$+ \left[{}^8C_6 \left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0\right]$$

$= 0.02194 + 0.1536 + 0.6145 + 1.5363$

$+ 2.4581 + 2.4581$

$P(x \geq 1) = 7.24254$

AI 2008, AI 2007 Pg = 285

**Q4(C)** Differentiate between conditional and marginal Probability.

# Conditional Probability

The sample space for an experiment must often be changed when some additional information pertaining to the outcom of the experiment is received. the effect of such information is to reduce the sample space by excluding some outcomes as being impossible which before reciving the information were belived possible. The probabilities associated with such a reduced sample space are called Conditional Probabilities.

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# Marginal Probability

From the joint probability function for (X,Y) we can obtain the individual Probability function of X and Y. such induvidual Probablity function are called marginal Probability.

# SECTION II

## Q5(a) Show that for binomial Distribution

$$\mu_{r+1} = pq\left(nr\mu_{r-1} + \frac{\partial \mu_r}{\partial p}\right)$$

## Solution :-

See Paper A/2010 Q2 (b) at Page "249"

* ——— *

A/2008

## Q5(b) Find the moment generating function of the normal distribution also find its Cumulants ; $K_2$, $K_4$ and variance.

## Solution :-

Solved by
Labeeb Ahmad

Let the r.v X be $N(\mu, \sigma^2)$ then the m.g.f of X with respect to origin is given by

$$M_o(t) = E(e^{tx}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} dx$$

Let

$$z = \frac{x-\mu}{\sigma} \quad \text{so that} \quad dx = \sigma dz$$

Then $M_0(t) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-z^2/2} \cancel{\sigma} dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu} \cdot e^{t\sigma z} e^{-z^2/2} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z - z^2/2} \cdot dz$$

Solved by Labeeb Ahmad

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z^2 - 2tz\sigma + t^2\sigma^2 - t^2\sigma^2]} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-t\sigma)^2 - t^2\sigma^2]} \cdot dz$$

Solved by Labeeb Ahmad

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} \cdot e^{\frac{t^2\sigma^2}{2}} \cdot dz$$

$$= \frac{e^{\mu t} \cdot e^{\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} \cdot dz$$

Solved by Labeeb Ahmad

$$M_0(t) = \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \cdot I_1 \quad \text{——— (1)}$$

where $I_1 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} . dz$

put $\frac{(z-t\sigma)^2}{2} = \theta$

$(z-t\sigma)^2 = 2\theta$

Differentiate both side

$\not{2}(z-t\sigma) . dz = \not{2} . d\theta$

$dz = \frac{d\theta}{z-t\sigma} \Rightarrow \frac{d\theta}{\sqrt{2\theta}}$

And

$\begin{cases} \theta \longrightarrow \infty & \text{when } z \longrightarrow \infty \\ \theta \longrightarrow -\infty & \text{when } z \longrightarrow -\infty \end{cases}$

putting these in $I_1$

$I_1 = \int_{-\infty}^{\infty} e^{-\theta} . \frac{d\theta}{\sqrt{2\theta}}$

$= \frac{2}{\sqrt{2}} \int_{0}^{\infty} e^{-\theta} . \theta^{-1/2} . d\theta$

$= \sqrt{2} \int_{0}^{\infty} \theta^{1/2-1} e^{-\theta} . d\theta$

$= \sqrt{2} \, \Gamma(1/2)$

$= \sqrt{2} . \sqrt{\pi}$

$I_1 = \sqrt{2\pi}$

**Solved by**
**Labeeb Ahmad**

Putting the value of $I_1$ in (1)

$$M_0(t) = \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \sqrt{2\pi}$$

$$= e^{\mu t + \frac{t^2\sigma^2}{2}}$$

The m.g.f with respect to $\mu$ is

$$M_\mu(t) = E[e^{t(x-\mu)}] = e^{-\mu t} E e^{(tx)}$$

$$= e^{-\mu t}\left[e^{\mu t + \frac{t^2\sigma^2}{2}}\right]$$

$$= e^{\frac{t^2\sigma^2}{2}}$$

Solved by
Labeeb Ahmad

As we know

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \frac{\theta^n}{n!}$$

There for

$$e^{\frac{t^2\sigma^2}{2}} = 1 + \frac{\left[\frac{1}{2}t^2\sigma^2\right]}{1!} + \frac{\left[\frac{t^2\sigma^2}{2}\right]^2}{2!} + \cdots \frac{\left[\frac{t^2\sigma^2}{2}\right]^n}{n!}$$

Hence $\mu_{2n+1} = 0$

$$\mu_{2n} = \text{Co-efficient of } \frac{t^2}{(2n)!}$$

Solved by
Labeeb Ahmad

$$= \left[\frac{\sigma^2}{2}\right]^n \frac{(2n)!}{n!}$$

$$= \frac{\sigma^{2n}}{2^n} \frac{(2n)!}{n!}$$

$$= \frac{\sigma^{2n}}{2^n . n!} \left[ 2n(2n-1)(2n-2)(2n-3) \cdots 5.4.3.2.1 \right]$$

$$= \frac{\sigma^{2n}}{2^n . n!} \left[ \left\{ 2n(2n-2)(2n-4) \cdots 4.2 \right\} \left\{ (2n-1)(2n-3) \cdots 5.3.1 \right\} \right]$$

$$= \frac{\sigma^{2n}}{2^n . n!} \left[ \left\{ 2n(2)(n-1).2(n-2) \cdots 4.2 \right\} \left\{ (2n-1)(2n-3 \cdots 5.3.1 \right\} \right]$$

$$= \frac{\sigma^{2n}}{\cancel{2^n . n!}} \left[ \cancel{\left\{ 2^n . n! \right\}} \left\{ (2n-1)(2n-3) \cdots 5.3.1 \right\} \right]$$

$$= 1.3.5 \cdots (2n-1)\, \sigma^{2n} \quad \forall\, n \geq 0$$

The cumulants generation function is given by

$$K(t) = \ln M_0(t) = \ln \left[ e^{\mu t + \frac{t^2 \sigma^2}{2}} \right]$$

Solved by
Labeeb Ahmad

$$= e^{\mu t + \frac{t^2 \sigma^2}{2}} . \ln . e$$

$$= \mu t + \frac{t^2 \sigma^2}{2}$$

thus $K_1 = \mu$ & $K_2 = \sigma^2$ and $K_r = 0 \; \forall\, r \geq 3$

A|2008



**Q6(a)** If X is an F-Distribution random variable m and n degree of freedom Then find mean & variance of X.

**Solution:-** see paper A|2012 Q8(b) At page "206"

**Solved by Labeeb Ahmad**

A|2008

**Q6(b)** If X is t-distribution r.v with n degree of freedom show that $1/(1+x^2/n)$ has Beta Distribution.

**Solution:-** see paper A|2012 Q8(a) at page "204"

A|2008, A|2006

**Q7(a)** If $X_1, X_2, ---- X_n$ are independent and identically Distributed gamma variates then what will be the distribution of $x_1 + x_2 + x_3$ —— $x_n$ ?

Pg = 292

**Solution** :- The m.g.f of $\gamma(n_1)$ variate with respect to origin is

$$M_0(t) = (1-t)^{-n_1}$$

Similarly, the m.g.f of $\gamma(n_2)$ variate is

$$M_0(t) = (1-t)^{-n_2}$$

Now the m.g.f of the sum is equal to the product of their m.g.f's.

That is

$$M_{x_1+x_2+x_3 \text{———} x_n} = E[e^{t(x_1+x_2+x_3 \text{———} x_n)}]$$

$$= E[e^{tx_1}] \cdot E[e^{tx_2}] \text{———} E[e^{tx_n}]$$

$$= M_{x_1}(t) \cdot M_{x_2}(t) \text{———} M_{x_n}(t)$$

where

$$x_1 \sim \frac{1}{\Gamma(n_1)} x^{n_1-1} \cdot e^{-x} \cdot dx$$

Solved by
Labeeb Ahmad

$$x_2 \sim \frac{1}{\Gamma(n_2)} x^{n_2-1} \cdot e^{-x} \cdot dx$$

| | | |

$$x_n \sim \frac{1}{\Gamma(n_n)} x^{n_n-1} e^{-x} \cdot dx$$

Thus $M_{x_1+x_2+x_3 \text{——} x_n} = (1-t)^{-n_1} \cdot (1-t)^{-n_2} --- \cdot (1-t)^{-n_n}$
$= (1-t)^{-(n_1+n_2+ \text{——} n_n)}$

But this is a m.g.f of a
$\gamma(n_1+n_2+n_3+ \text{——} n_n)$ variaties

Hence the Sum of n independent Gamma distributions with parameter $n_1, n_2$ —— $n_n$ is a Gamma distributed with parameter $(n_1+n_2$ —— $n_n)$

---

**Solved by**
**Labeeb Ahmad**

**Q7(b)** Derive $\chi^2$ distribution is a limiting Form of F-distribution also find variance.

**Solution:-** see paper A/2016 Q8(a) at Page "33"
For Variance see page "210"

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**Q8(b)** If $x_1, x_2$ & $x_3$ are not Correlated and $\bar{x}_i = 0$ for $i = 1, 2, 3$, they have equal variance then show that the Co-efficient of correlation b/w $x_1+x_2$ and $x_1+x_3$ is 1/2. {Solution} — see Paper A/2012 Q7(b) at Page 202

A/2008 , A/2004 Pg = 294

**Q8(a)** If $x_1$ and $x_2$ are independent random variables with mean 5 and 10 and variances 4 and 9 respectively. Find the Co-efficient of Co-rrelation between $y_1$ and $y_2$ where;

$$y_1 = 3x_1 + 4x_2 \quad \text{and} \quad y_2 = 3x_1 - x_2$$

**Solution:-**

$$r_{y_1 y_2} = \frac{\sum (y_1 - \bar{y}_1)(y_2 - \bar{y}_2)}{\sqrt{\sum (y_1 - \bar{y}_1)^2 \sum (y_2 - \bar{y}_2)}} \quad \text{——} (1)$$

Now $y_1 - \bar{y}_1 = (3x_1 + 4x_2) - (3\bar{x}_1 + 4\bar{x}_2)$

$$= 3(x_1 - \bar{x}_1) + 4(x_2 - \bar{x}_2)$$

$$\sum (y_1 - \bar{y}_1)^2 = \sum \left[3(x_1 - \bar{x}_1) + 4(x_2 - \bar{x}_2)\right]^2$$

$$= \sum \left[9(x_1 - \bar{x}_1)^2 + 16(x_2 - \bar{x}_2)^2 + 24(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)\right]$$

$$= 9\sum (x_1 - \bar{x}_1)^2 + 16\sum (x_2 - \bar{x}_2)^2 + 24\sum (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)$$

As $Var\, x_1 = 4$

$Var\, x_2 = 9$

**Solved by**
**Labeeb Ahmad**

$$= 9n(4) + 16\, n(9) + 0$$

Because $x_1$ & $x_2$ are not corelated

$$\Sigma(y_1-\bar{y}_1)^2 = 36n + 144n = 180n$$

And

$$y_2-\bar{y}_2 = (3x_1 - x_2) - (3\bar{x}_1 - \bar{x}_2)$$
$$= 3(x_1-\bar{x}_1) - (x_2-\bar{x}_2)$$
$$\Sigma(y_2-\bar{y}_2)^2 = \Sigma\left[3(x_1-\bar{x}_1)-(x_2-\bar{x}_2)\right]^2$$
$$= \Sigma\left[9(x_1-\bar{x}_1)^2 + (x_2-\bar{x}_2)^2 - 6(x_1-\bar{x}_1)(x_2-\bar{x}_2)\right]$$
$$= 9\Sigma(x_1-\bar{x}_1)^2 + \Sigma(x_2-\bar{x}_2)^2 - 6\Sigma(x_1-\bar{x}_1)(x_2-\bar{x}_2)$$
$$= 9n(4) + n\cdot(9)$$
$$= 36n + 9n$$
$$\Sigma(y_2-\bar{y}_2)^2 = 45n$$

**Solved by Labeeb Ahmad**

And $\Sigma(y_1-\bar{y}_1)(y_2-\bar{y}_2)$

$$= \Sigma\left[3(x_1-\bar{x}_1)+4(x_2-\bar{x}_2)\right]\left[3(x_1-\bar{x}_1)-(x_2-\bar{x}_2)\right]$$
$$= \Sigma\left[9(x_1-\bar{x}_1)^2 - 3(x_1-\bar{x}_1)(x_2-\bar{x}_2) + 12(x_1-\bar{x}_1)(x_2-\bar{x}_2) - 4(x_2-\bar{x}_2)^2\right]$$
$$= 9\Sigma(x_1-\bar{x}_1)^2 + 9\Sigma(x_1-\bar{x}_1)(x_2-\bar{x}_2) - 4\Sigma(x_2-\bar{x}_2)^2$$
$$= 9n(4) + 0 - 4n(9)$$
$$= 36n - 36n = 0$$

Substituting these values in (a)

$$r_{y_1y_2} = \frac{0}{\sqrt{(180n)(45n)}} = 0$$

Hence $y_1, y_2$ are not correlated

## Q9(a) The data have been obtained on X = rain fall (inches) and Y = yield of wheat (bushels) per acre. Fit a least square line to predict the yield of wheat.

| X | 12.9 | 7.2 | 11.3 | 18.6 | 8.8 | 10.3 | 15.9 | 13.1 |
|---|---|---|---|---|---|---|---|---|
| Y | 63 | 29 | 52 | 81 | 42 | 71 | 54 | 45 |

**Solution:-** The estimated least square line Y on X is

$$\hat{Y} = a + bX$$

and two normal equations are

$$\sum Y = na + b\sum X$$

$$\sum XY = a\sum X + b\sum X^2$$

**Solved by Labeeb Ahmad**

To compute the necessary summation use arrange the data.

| X | Y | XY | $X^2$ | X | Y | XY | $X^2$ |
|---|---|---|---|---|---|---|---|
| 12.9 | 63 | 812.7 | 166.41 | 8.8 | 42 | 369.6 | 77.44 |
| 7.2 | 29 | 208.8 | 51.84 | 10.3 | 71 | 731.3 | 106.09 |
| 11.3 | 52 | 587.6 | 127.69 | 15.9 | 54 | 858.6 | 252.81 |
| 18.6 | 81 | 1506.6 | 345.96 | 13.1 | 45 | 589.5 | 171.61 |

So $\sum X = 98.1$ , $\sum Y = 437$

$\sum XY = 5664.7$ , $\sum X^2 = 1299.85$

$$\bar{x} = \frac{\sum x}{n} = \frac{98.1}{8} = 12.26$$

$$\bar{y} = \frac{\sum y}{n} = \frac{437}{8} = 54.62$$

$$b = \frac{n\sum XY - \sum X \cdot \sum Y}{n\sum X^2 - (\sum X)^2}$$

$$b = \frac{8(5664.7) - (98.1)(437)}{8(1299.85) - (98.1)^2} = \frac{2447.9}{775.19}$$

$$b = 3.15$$

As $\bar{Y} = a + b\bar{X}$ or $a = \bar{Y} - b\bar{X}$

$$a = 54.62 - (3.15)(12.26)$$

$$a = 54.62 - 38.619$$

$$a = 16$$

The estimated least square line is

$$\hat{Y} = a + bX$$

$$\hat{Y} = 16 + 3.15X$$

**Solved by**
**Labeeb Ahmad**

A/2008 , A/2004 Pg = 298

## Q9(b) Write short note on

(i) Linear Regression

(ii) Method of Least Square

(iii) The Distribution of Mean for finite Population

# Solution:-

(i)

## Linear Regression

The relation between the expected value of the dependent variable and the independent variable is called a regression relation.

And when the dependence is represented by a straight line equation, the regression is said to be linear regression.

(ii)

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## Method of Least Square

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The principle of least square consists of determining the values of unknown parameters that will be minimize the sum of squares of errors or residuals) where errors are defined as differences between observed and corresponding values predicted or estimated by the fitted Model equation.

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A/2000, A/2002 Pg = 300

# Q F-Distribution

The ratio of the independent $\chi^2$ random variable divided by their respectively degree's of freedom has F-Distribution.

**Proof:-** Let U and V be two r-v have ($\chi^2$ distribution) with n and m degree of freedom respectively.

The density function of U & V is given as

$$f_1(U) = \frac{e^{-\frac{u}{2}} U^{\frac{n}{2}-1}}{2^{n/2}\Gamma(n/2)} \qquad 0 < u < \infty$$

$$f_2(V) = \frac{e^{-\frac{v}{2}} V^{\frac{m}{2}-1}}{2^{m/2}\Gamma(m/2)}$$

**Solved by Labeeb Ahmad**

Now

$$f_{12}(U,V) = \left[\frac{e^{-\frac{U}{2}} U^{\frac{n}{2}-1}}{2^{n/2}\Gamma(n/2)}\right]\left[\frac{e^{-\frac{V}{2}} V^{\frac{m}{2}-1}}{2^{m/2}\Gamma(m/2)}\right]$$

$$= \frac{e^{-\frac{U}{2}} \cdot e^{-\frac{V}{2}} U^{\frac{n}{2}-1} V^{\frac{m}{2}-1}}{2^{\frac{n+m}{2}}\Gamma(n/2)\Gamma(m/2)}$$

Now let $F = \dfrac{U/n}{V/m} = \dfrac{U}{n} \times \dfrac{m}{V}$

$\Rightarrow U = V\dfrac{n}{m}F$ and $v = V$

Now we find the density function of $F$ and $V$

Jacobian Transformation:
If we chang "n" variables to anothe "n" variables then we apply Jacobian of transformation

For example

$f(r,\theta) = f(x,y)\,|J|$

and $J\dfrac{(x,y)}{(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$

Solved by
Labeeb Ahmad

Here we chang $U, V$ into $F, V$

There for we apply Jacobian Transformation

As $U = V\frac{n}{m}F$ , $V = V$

$$\frac{\partial U}{\partial F} = V\frac{n}{m} \qquad \frac{\partial V}{\partial F} = 0$$

$$\frac{\partial U}{\partial V} = \frac{n}{m}F \qquad \frac{\partial V}{\partial V} = 1$$

$$|J| = \begin{vmatrix} \frac{\partial U}{\partial F} & \frac{\partial U}{\partial V} \\ \frac{\partial V}{\partial F} & \frac{\partial V}{\partial V} \end{vmatrix} = \begin{vmatrix} V\frac{n}{m} & 0 \\ \frac{n}{m}F & 1 \end{vmatrix} = V\frac{n}{m}$$

$$U = V\frac{n}{m}F \qquad V$$

$$f(F,V) = f(U, V)\,|J| \quad \text{——} \ (i)$$

$$f(F,V) = \frac{e^{-\frac{1}{2}\left[V\frac{n}{m}F + V\right]} \left(\frac{Vn F}{m}\right)^{\frac{n}{2}-1} V^{\frac{m}{2}-1}}{2^{\frac{n+m}{2}}\,\Gamma(n/2)\,\Gamma(m/2)} \cdot \frac{Vn}{m}$$

**Solved by Labeeb Ahmad**

Put in (i)

$$f(F,V) = \frac{e^{-\frac{V}{2}\left[\frac{n}{m}F + 1\right]} \left(\frac{Vn}{m}\right)^{\frac{n}{2}} F^{\frac{n}{2}-1} V^{\frac{m}{2}-1}}{2^{\frac{n+m}{2}}\,\Gamma(n/2)\,\Gamma(m/2)}$$

$$f(F,V) = \frac{e^{-\frac{V}{2}\left[\frac{n}{m}F + 1\right]} (n/m)^{n/2} F^{\frac{n}{2}-1} V^{\frac{n}{2}+\frac{m}{2}-1}}{2^{\frac{n+m}{2}}\,\Gamma(n/2)\,\Gamma(m/2)}$$

Now we find the density function of F

$$f(F) = \int_0^\infty f(F,v)\cdot dv$$

$$= \int_0^\infty \frac{(n/m)^{n/2} F^{\frac{n}{2}-1} e^{-\frac{v}{2}\left(\frac{nF}{m}+1\right)} v^{\frac{n}{2}+\frac{m}{2}-1}}{2^{\frac{n+m}{2}} \Gamma(n/2) \Gamma(m/2)} dv$$

$$= \frac{(n/m)^{\frac{n}{2}} F^{\frac{n}{2}-1}}{2^{\frac{n+m}{2}} \Gamma(n/2) \Gamma(m/2)} \int_0^\infty e^{-\frac{v}{2}\left(\frac{nF}{m}+1\right)} v^{\frac{n}{2}+\frac{m}{2}-1} dv \quad \text{———} (2)$$

Take $\int_0^\infty e^{-\frac{v}{2}\left(\frac{nF}{m}+1\right)} v^{\frac{n}{2}+\frac{m}{2}-1} dv$

Put $\theta = \frac{v}{2}\left(\frac{nF+m}{m}\right)$

$$d\theta = \frac{dv}{2}\left(\frac{nF+m}{m}\right)$$

**Solved by Labeeb Ahmad**

$$\frac{2m\,d\theta}{nF+m} = dv$$

And $\begin{cases} \theta \to \infty & \text{when } v \to \infty \\ \theta \to 0 & \text{when } v \to 0 \end{cases}$

$$\int_0^\infty e^{-\frac{v}{2}\left(\frac{nF+m}{m}\right)} v^{\frac{n}{2}+\frac{m}{2}-1} \cdot dv$$

$$= \int_0^\infty e^{-\theta} \left(\frac{2m\theta}{nF+m}\right)^{\frac{n}{2}+\frac{m}{2}-1} \left(\frac{2m}{nF+m}\right) d\theta$$

$$= \int_0^\infty e^{-\theta} \left(\frac{2m}{nF+m}\right)^{\frac{n+m}{2}} \theta^{\frac{n}{2}+\frac{m}{2}-1} \cdot d\theta$$

$$= \left(\frac{2m}{nF+m}\right)^{\frac{n+m}{2}} \int_0^\infty e^{-\theta} \theta^{\frac{n}{2}+\frac{m}{2}-1} \cdot d\theta$$

$$= \left(\frac{2m}{nF+m}\right)^{\frac{n+m}{2}} \Gamma\left(\frac{n}{2}+\frac{m}{2}\right)$$

put in (2)

$$f(F) = \frac{(n/m)^{\frac{n}{2}} F^{\frac{n}{2}-1}}{2^{\frac{n+m}{2}} \Gamma(n/2)\Gamma(m/2)} (2m/nF+m)^{\frac{n+m}{2}} \Gamma\left(\frac{n}{2}+\frac{m}{2}\right)$$

$$= \frac{\Gamma\left(\frac{n}{2}+\frac{m}{2}\right)}{\Gamma(n/2)\Gamma(m/2)} (n/m)^{n/2} \frac{F^{\frac{n}{2}-1}}{2^{\frac{n+m}{2}}} \frac{2^{\frac{n+m}{2}} m^{\frac{n}{2}+\frac{m}{2}}}{(nF+m)^{\frac{n+m}{2}}}$$

**Solved by**
**Labeeb Ahmad**

$$f(F) = \frac{1}{\beta\left(\frac{n}{2}, \frac{m}{2}\right)} (n/m)^{n/2} F^{\frac{n}{2}-1} \frac{m^{\frac{n+m}{2}}}{m^{\frac{n+m}{2}} \left[1 + \frac{n}{m}F\right]^{\frac{n+m}{2}}}$$

$$f(F) = \frac{(n/m)^{\frac{n}{2}}}{\beta\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot F^{\frac{n}{2}-1} \cdot \frac{1}{\left(1 + \frac{n}{m}F\right)^{\frac{n+m}{2}}}$$

**Solved by**
**Labeeb Ahmad**

# SPECIAL Assignment

## $\Gamma(1/2) = \sqrt{\pi}$

**Proof :-**

The gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \cdot dx$$

Making the substitution $x = u^2$

$$\Gamma(\alpha) = 2\int_0^\infty u^{2\alpha-1} e^{-u^2} du$$

A special value of gamma function can be derived when

$2\alpha - 1 = 0$

or $\alpha = 1/2$

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2} du$$

To derive the value for $\Gamma(1/2)$

Solved by
Labeeb Ahmad

First, the values $\Gamma(1/2)$ is squared.
Second the squared value is rewritten as double integral.
Third, the double integral is evaluated by

$$\left[\Gamma(1/2)\right]^2 = \left[\Gamma(1/2)\right]\left[\Gamma(1/2)\right]$$

$$= \left[2\int_0^\infty e^{-u^2}\cdot du\right]\left[2\int_0^\infty e^{-v^2}\, dv\right]$$

$$= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\, du\cdot dv$$

The region R which defines the first quadrant, is the region of integration for integral (1)

The bivariate transformation

$u = r\cos\theta$

$v = r\sin\theta$

will be transform the integral problem for cartesian quadrant.

Solved by
Labeeb Ahmad

The Jacobian of the transformation is

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

Hence

$$\left[\Gamma(1/2)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\, dv.du$$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r.dr.d\theta$$

$$= 4\left[\int_0^{\pi/2} d\theta\right]\left[\int_0^\infty e^{-r^2}\, r.dr\right]$$

$$= 4\left[\theta \Big|_0^{\pi/2}\right]\left[\int_0^\infty e^{-r^2}\, r.dr\right]$$

put $u = -r^2$

$du = -2r.dr$

$$\begin{cases} u \longrightarrow 0 \text{ when } r \longrightarrow 0 \\ u \longrightarrow \infty \text{ when } r \longrightarrow \infty \end{cases}$$

Solved by
Labeeb Ahmad

$= 4\left[\frac{\pi}{2}\right]\left[-\frac{1}{2}\int_0^{-\infty} e^{u}.du\right]$

$= 2\pi\left[-\frac{1}{2}\left(e^{u}\Big|_0^{-\infty}\right)\right.$

$= 2\pi\left[-\frac{1}{2}\left(e^{-\infty} - e^{0}\right)\right]$

$= 2\pi\left[-\frac{1}{2}(0-1)\right]$

$= 2\pi\,(1/2)$

$= \pi$

Finally since $e^{-u^2} > 0$ for all $u \neq 0$

Then $\Gamma(1/2) \neq 0$

Hence $\left[\Gamma(1/2)\right]^2 = \pi$

$\left[\Gamma(1/2)\right] = \sqrt{\pi}$

$\Gamma(1/2) = \sqrt{\pi}$

Hence Prove

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0341-7224005

**Solved by**
**Labeeb Ahmad**

**Q6(a)** If X and Y have zero mean same variance $\sigma^2$ and zero Correlation, then show that
$X\cos\alpha + Y\sin\alpha$ and $X\sin\alpha - Y\cos\alpha$
have same variance $\sigma^2$ and zero Correlation.

Solved by Labeeb Ahmad

## Solution :-

let $U = X\cos\alpha + Y\sin\alpha$
$V = X\sin\alpha - Y\cos\alpha$

> Note :- $\alpha$ is a angle measure in Radian. There for
> $\sin\alpha \in [-1,1]$
> And $\cos\alpha \in [-1,1]$

As
$$r_{UV} = \frac{\sum(U-\bar{U})(V-\bar{V})}{\sqrt{\sum(U-\bar{U})^2 \sum(V-\bar{V})^2}} \quad \text{--- (1)}$$

$$U - \bar{U} = (X\cos\alpha + Y\sin\alpha) - (\bar{X}\cos\alpha + \bar{Y}\sin\alpha)$$
$$= (X-\bar{X})\cos\alpha + (Y-\bar{Y})\sin\alpha$$

$V - \bar{V} = (X \sin\alpha - Y \cos\alpha) - (\bar{X} \sin\alpha - \bar{Y} \cos\alpha)$

$= (X - \bar{X}) \sin\alpha - (Y - \bar{Y}) \cos\alpha$

And

$\sum (U - \bar{U})(V - \bar{V}) = \sum \Big( \left[ (X - \bar{X}) \cos\alpha + (Y - \bar{Y}) \sin\alpha \right]$

**Solved by**
**Labeeb Ahmad**

$\left[ (X - \bar{X}) \sin\alpha - (Y - \bar{Y}) \cos\alpha \right] \Big)$

$= \sum \Big( (X - \bar{X})^2 \sin\alpha \cdot \cos\alpha$

$- (X - \bar{X})(Y - \bar{Y}) \cos^2\alpha + (Y - \bar{Y})(X - \bar{X}) \sin^2\alpha$

$- (Y - \bar{Y})^2 \sin\alpha \cdot \cos\alpha \Big)$

$= \sin\alpha \cdot \cos\alpha \cdot \sum (X - \bar{X})^2 - \cos^2\alpha \sum (X - \bar{X})(Y - \bar{Y})$

$+ \sin^2\alpha \sum (X - \bar{X})(Y - \bar{Y}) - \sin\alpha \cos\alpha \sum (Y - \bar{Y})^2$

Becaus $X_1$ and $X_2$ are uncorelated

There for $\sum (X - \bar{X})(Y - \bar{Y}) = 0$

$= \sin\alpha \cdot \cos\alpha \cdot n \cdot \frac{\sum (X - \bar{X})^2}{n} - 0$

$+ 0 - \sin \cdot \cos\alpha \cdot n \frac{\sum (Y - \bar{Y})^2}{n}$

$= n \sin\alpha \cdot \cos\alpha \cdot Var(X) - n \sin\alpha \cdot \cos\alpha \, Var(Y)$

As $Var(X) = Var(Y) = \sigma^2$

$= n \sin\alpha \cdot \cos\alpha \cdot \sigma^2 - n \sin\alpha \cdot \cos\alpha \, \sigma^2$

$= 0$

Put this in (1) $r_{UV} = 0$

Hence $U$ and $V$ has zero Correlation.

**Q6(b)** Find the line of regression Y on X and X on Y for the following data.

| X | 1 | 2 | 3 | 4 | 6 |
|---|---|---|---|---|---|
| Y | 1 | 4 | 9 | 16 | 18 |

**Solution:-** The estimated regression line Y on X is

$$\hat{Y} = a + b_{yx}X$$

and the two normal equations are

$$\Sigma Y = na + b\Sigma X$$

$$\Sigma XY = a\Sigma X + b\Sigma X^2$$

To Compute the necessary Summation arrange the data.

| X | Y | XY | $X^2$ | $Y^2$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 8 | 4 | 16 |
| 3 | 9 | 27 | 9 | 81 |
| 4 | 16 | 64 | 16 | 256 |
| 6 | 18 | 108 | 36 | 324 |
| $\Sigma X = 16$ | $\Sigma Y = 48$ | $\Sigma XY = 208$ | $\Sigma X^2 = 66$ | $\Sigma Y^2 = 678$ |

Solved by
Labeeb Ahmad

$\bar{X} = \frac{\Sigma X}{n} = \frac{16}{5} = 3.2$

$\bar{Y} = \frac{\Sigma Y}{n} = \frac{48}{5} = 9.6$

$b_{yx} = \frac{n \Sigma XY - (\Sigma X)(\Sigma Y)}{n \Sigma X^2 - (\Sigma X)^2}$

$b_{yx} = \frac{(5)(208) - (16)(48)}{(5)(66) - (16)^2}$

$b_{yx} = \frac{1040 - 768}{330 - 256}$

$b_{yx} = \frac{272}{74}$

**Solved by**
**Labeeb Ahmad**

$b_{yx} = 3.67$

As $a = \bar{Y} - b_{yx} \bar{X}$

$a = 9.6 - (3.67)(3.2)$

$a = 9.6 - 11.744$

$a = -2.144$

Thus estimated regression line is

$\hat{Y} = a + b_{yx} \bar{X}$

$= -2.144 + 3.67 X$

where b is regression which shows value of Y increases by 3.67 units for a a unit increase by x.

The estimated regression line X on Y is

$$\hat{X} = a' + b_{XY} Y$$

$$b_{XY} = \frac{n\sum XY - \sum X \cdot \sum Y}{n\sum Y^2 - (\sum Y)^2}$$

$$= \frac{5(208) - (16)(48)}{5(678) - (48)^2}$$

$$= \frac{1040 - 768}{3390 - 2304}$$

$$= \frac{272}{1086}$$

**Solved by**
**Labeeb Ahmad**

$$b_{XY} = 0.2504$$

And $a' = \bar{X} - b_{XY}\bar{Y}$

$$a' = (3.2) - (0.2504)(9.6)$$

$$a' = 3.2 - 2.4044$$

$$a' = 0.7955$$

Thus the estimated regration line X on Y is

$$\hat{X} = a' + b_{XY} Y$$

$$\hat{X} = 0.7955 + (0.2504) Y$$

A|2006 Pg = 315

# Q4(b) Show that

$$\frac{\beta(m+2, n-2)}{\beta(m,n)} = \frac{m(m+1)}{(n-1)(n-2)}$$

## Solution:

Take $\dfrac{\beta(m+2, n-2)}{\beta(m,n)}$ —— (i)

Take $\beta(m+2, n-2) = \dfrac{\Gamma(m+2) \cdot \Gamma(n-2)}{\Gamma(m+2+n-2)}$

Solved by
Labeeb Ahmad

$$= \frac{\Gamma(m+2) \cdot \Gamma(n-2)}{\Gamma(m+n)} \quad (2)$$

As $\Gamma(m+2) = (m+1)\Gamma(m+1)$

$= (m+1) m \Gamma(m)$

And $\Gamma(n-2) = \dfrac{n-2}{n-2} \Gamma(n-2)$

$= \dfrac{\Gamma(n-1)}{(n-2)}$

$= \dfrac{n-1}{n-1} \dfrac{\Gamma(n-1)}{(n-2)} = \dfrac{\Gamma(n)}{(n-1)(n-2)}$

Putting these in (2)

$$= \frac{m(m+1)\Gamma(m) \cdot \dfrac{\Gamma(n)}{(n-1)(n-2)}}{\Gamma(m+n)}$$

$$= \frac{m(m+1)}{(n-1)(n-2)} \cdot \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{m(m+1)}{(n-1)(n-2)} \cdot B(m,n)$$

put this (i)

$$\frac{B(m+2,\ n-2)}{B(m,n)} = \frac{m(m+1)}{(n-1)(n-2)} \cdot \frac{B(m,n)}{B(m,n)}$$

$$= \frac{m(m+1)}{(n-1)(n-2)}$$

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Ans

A/2003



**Q4(b)** Show that the Beta Distribution

$$f(x) = \frac{1}{\beta(\alpha+1, \lambda+1)} \cdot x^{\alpha}(1-x)^{\lambda} \qquad 0 \le x \le 1$$

becomes the Gamma distribution as the parameter $\lambda \to \infty$

## Solution :-

$$f(x) = \frac{1}{\beta(\alpha+1, \lambda+1)} x^{\alpha}(1-x)^{\lambda}$$

put $\alpha = \alpha' - 1$

$\lambda = \beta' - 1$

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And $\lambda \to \infty$ then $\beta' \to \infty$

$$f(x) = \frac{1}{\beta(\alpha', \beta')} \cdot x^{\alpha'-1}(1-x)^{\beta'-1}$$

As

$$d\,f(x) = \frac{1}{\beta(\alpha', \beta')} x^{\alpha'-1}(1-x)^{\beta'-1} dx$$

let $z = \beta' x \Rightarrow dz = \beta' dx$

$$dx = \frac{dz}{\beta'}$$

$$x = \frac{z}{\beta'}$$

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Then $df(x) = \dfrac{1}{\beta(\alpha', \beta')} \cdot \left(\dfrac{z}{\beta'}\right)^{\alpha'-1}\left(1-\dfrac{z}{\beta'}\right)^{\beta'-1}\left(\dfrac{1}{\beta'}\right)dz$

$= \dfrac{\Gamma(\alpha'+\beta')}{\Gamma(\alpha') * \Gamma(\beta')} (z)^{\alpha'-1}\left(\dfrac{1}{\beta'}\right)^{\alpha'-1}\left(1-\dfrac{z}{\beta'}\right)^{\beta'}\left(1-\dfrac{z}{\beta'}\right)^{-1}\left(\dfrac{1}{\beta'}\right)dz$

$= \dfrac{\Gamma(\alpha'+\beta')}{\Gamma(\alpha') * \Gamma(\beta')} \dfrac{z^{\alpha'-1}}{\beta'^{\alpha'}}\left[\left(1-\dfrac{z}{\beta'}\right)^{\frac{\beta'}{z}}\right]^{z}\left(1-\dfrac{z}{\beta'}\right)^{-1}dz$

Now applying $\lim \beta' \to \infty$

$\lim_{\beta'\to\infty} df(x) = z^{\alpha'-1} \lim_{\beta'\to\infty} \dfrac{\Gamma(\alpha'+\beta')}{\Gamma(\alpha')\cdot\Gamma(\beta')\,\beta'^{\alpha'}}$

$\lim_{\beta\to\infty}\left[\left(1-\dfrac{z}{\beta'}\right)^{-\frac{\beta'}{z}}\right]^{-z} \cdot \lim_{\beta'\to\infty}\left[1-\dfrac{z}{\beta'}\right]^{-1} dz$ ——— (1)

Taking $\lim_{\beta'\to\infty} \dfrac{\Gamma(\alpha'+\beta')}{\Gamma(\alpha')\cdot\Gamma(\beta')\;\beta'^{\alpha'}}$

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$= \lim_{\beta'\to\infty} \dfrac{(\alpha'+\beta'-1)!}{\Gamma(\alpha')\,\Gamma(\beta')\,\beta'^{\alpha'}}$

$= \lim_{\beta'\to\infty} \dfrac{(\alpha'+\beta'-1)(\alpha'+\beta'-2)\cdots(\alpha'+\beta'-\alpha')\,\Gamma(\alpha'+\beta'-\alpha')}{\Gamma(\alpha')\;\Gamma(\beta')\;\beta'^{\alpha'}}$

$$\lim_{\beta'\to\infty} \frac{\beta'\left[1+\frac{\alpha'-1}{\beta'}\right]\beta'\left[1+\frac{\alpha'-2}{\beta'}\right]\cdots\cdots \cancel{\Gamma(\beta')}}{\Gamma(\alpha')\cdot\cancel{\Gamma(\beta')}\cdot\beta'^{\alpha'}}$$

$$\lim_{\beta'\to\infty} \frac{\cancel{(\beta')^{\alpha'}}\left[1+\frac{\alpha'-1}{\beta'}\right]\left[1+\frac{\alpha'-2}{\beta'}\right]\cdots\cdots}{\Gamma(\alpha')\cdot\cancel{(\beta')^{\alpha'}}}$$

$$= \frac{(1)(1)\cdot(1)\cdot(1)\cdots\cdots}{\Gamma(\alpha')}$$

$$= \frac{1}{\Gamma(\alpha')}$$

Now Taking $\lim_{\beta'\to\infty}\left[\left(1-\frac{z}{\beta'}\right)^{-\beta'/z}\right]^{-z}$

$$= e^{-z}$$

Taking $\lim_{\beta'\to\infty}\left[\left(1-\frac{z}{\beta}\right)^{-1}\right] = 1$

**Solved by**
**Labeeb Ahmad**

Putting these in (1)

$$\lim_{\beta'\to\infty} d\,f(x) = z^{\alpha'-1}\,\frac{1}{\Gamma(\alpha')}\,e^{-z}\,.dz$$

$$= \frac{1}{\Gamma(\alpha')}\,z^{\alpha'-1}\,e^{-z}\,.dz \qquad z>0$$

Now it becomes gamma distribution

Hence Proved

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